Some sufficient conditions for the Taketa inequality

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(Communicated by Masaki KASHIWARA, M.J.A., Oct. 15, 2013)

Abstract: In this study we have obtained some sufficient conditions for the Taketa inequality namely $dl(G) \leq |cd(G)|$ for finite solvable groups G.

Key words: Taketa inequality; character degrees; supersolvable groups.

1. Introduction. A long standing open problem in the character theory of finite solvable groups is whether the derived length $dl(G)$ of a solvable group G is bounded above by the cardinality of $cd(G)$, the set of irreducible character degrees of that group, i.e. whether the so-called Taketa inequality $dl(G) \leq |cd(G)|$ is true for every finite solvable group G. This inequality appeared first in the proof of the fact that all M-groups are solvable. This proof was given by Taketa by establishing that an M-group has to satisfy the Taketa inequality. The famous Isaacs-Seitz conjecture claims that the Taketa inequality is true not only for M-groups but for any finite solvable group. In the literature we know only some classes of solvable groups besides M-groups for which the conjecture is true. For example, T. R. Berger has shown that all finite groups of odd order satisfy the Taketa inequality [1]. In their paper, "Irreducible character degrees and normal subgroups" I. M. Isaacs and G. Knutson [5] have proved that if N is a normal nilpotent subgroup of G then $dl(N) \leq |cd(G|N)|$ where $cd(G|N)$ is the set of degrees of irreducible characters of G whose kernels do not contain N. They also remark that the inequality $dl(N) \leq |cd(G|N)|$ includes the Taketa inequality as a special case when N is replaced by G'. As a corollary, it turns out that they prove that $dl(G) \leq |cd(G)|$ when G' is nilpotent. Some of the other sufficient conditions refer to the cardinality of $cd(G)$. I. M. Isaacs has shown that the condition $|cd(G)| \leq 3$ is sufficient for the Taketa inequality [4] (or Corollary 12.6 and Theorem 12.15 of [6]). In his Ph.D. thesis, S. Garrison has obtained that $|cd(G)| = 4$ is another sufficient condition for the conjecture which is later generalized by I. M. Isaacs and Greg Knutson (see Theorem C of [5]). The last known sufficient condition for the Taketa inequality regarding the cardinality of the set of the irreducible character degrees is [7] due to Mark Lewis dealing with the case $|cd(G)| = 5$. The problem is still open for solvable groups with six irreducible character degrees.

Motivated by these results we obtain in this paper some further sufficient conditions for the conjecture.

2. Main theorems. We start with the following proposition.

Proposition 2.1. *Let G be a finite group and let N be a normal Hall subgroup of G. Suppose that both G/N' and N satisfy the Taketa inequality. Then G satisfies the Taketa inequality.*

The proof of this Proposition 2.1 is essentially the same as the proof of Lemma 12.16 of [6]. But for the sake of completeness and as a short reminder we repeat a condensed form of the proof here.

Proof. Let π be the set of primes dividing $|N|$. Since N is a normal Hall subgroup of G, $cd(N)$ is exactly the set of π-parts of the elements of $cd(G)$ and every degree in $cd(G/N')$ divides the index $|G : N|$ by Theorem 6.15 of [6]. This yields that $|cd(N)| + |cd(G/N')| - 1 \leq |cd(G)|$. Now we have $dl(G) \leq dl(G/N') + dl(N') \leq dl(G/N') + dl(N) - 1 \leq |cd(G/N')| + |cd(N)| - 1 \leq |cd(G)|$ as desired. □

As a corollary of this proposition, we give a generalization of the fact that supersolvable groups satisfy the Taketa inequality (see Theorem 6.22 of [6]).

Theorem 2.2. *Let G be a finite group and p be the smallest prime divisor of the order of G. If G has a normal p-complement then G satisfies the Taketa inequality.*

2000 Mathematics Subject Classification. Primary 20C15.
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doi: 10.3792/pjaa.89.103

Proof. Since all finite groups of odd order satisfy the Taketa inequality by [1], we may assume that the order of G is even so that $p = 2$. Let N be the normal 2-complement of G. Since the order of N is odd, N satisfies the Taketa inequality. Also, N/N' is an abelian normal subgroup of G/N' and the factor group is a 2-group. So G/N' is an M-group by Theorem 6.22, Theorem 6.23 of [6] and satisfies the Taketa inequality. Thus G itself satisfies the Taketa inequality by Proposition 2.1. $\square$

Corollary 2.3. *Let M be a normal subgroup of a group G, where M is supersolvable and G/M is a p-group where p is the smallest prime number dividing $|G|$. Then G satisfies the Taketa inequality.*

Proof. We know that M has a normal p-complement and since G/M is a p-group G has also a normal p-complement. So we are done by Theorem 2.2. $\square$

Corollary 2.4. *Let G be a rational group with supersolvable derived subgroup. Then G satisfies the Taketa inequality.*

Proof. Since G is rational and factor groups of a rational group are still rational, G/G' is a rational group which is also abelian. It is well known that only abelian rational groups are elementary abelian 2-groups. So we are done by Corollary 2.3. $\square$

Theorem 2.5. *Let N be a normal subgroup of a group G, where N has an abelian normal p-complement for some prime number p. Then $dl(N) \leq |cd(G|N)|$. In particular, if G' has an abelian normal p-complement, then $dl(G) \leq |cd(G)|$.*

Proof. We will induct on $|N|$. If $|N| = 1$, then $dl(N) = 0$ and the result holds. Assume $N > 1$. We have $dl(N) = 1 + dl(N') \leq 1 + |cd(G|N')| \leq |cd(G|N)|$, where the first inequality holds by the inductive hypothesis since $N' < N$, and the second inequality holds by Theorem 3.1 of [5]. To establish the second claim of the theorem, replace N with G'. $\square$

Let us consider the following condition for a solvable group G:

$$\pi(H') < \pi(H)$$

for every nontrivial Hall subgroup H of G. Under this condition, all Sylow subgroups of G are abelian and so G is an M-group by Theorem 6.23 of [6]. Thus the condition above is sufficient for the Taketa inequality. In the next theorem, we will provide a slightly weaker sufficient condition:

Theorem 2.6. *Let G be a solvable group. Assume that $\pi(H') < \pi(H)$ for every Hall subgroups H of G satisfying $2 \leq |\pi(H)|$. Then $dl(G) \leq |cd(G)|$.*

Proof. We will induct on the order of G. Since Taketa inequality holds for p-groups, we may assume that $2 \leq |\pi(G)|$. This starts the induction and also allows us to conclude $\pi(G') < \pi(G)$ by the fact that every group is a Hall subgroup of itself.

Thus there exists a prime number q dividing the order of G but fails to divide the order of G'. Thus G' is a q'-group and so a Hall q'-subgroup H of G contains G'. Since H is a Hall subgroup of G, the hypothesis is satisfied for H and so $dl(H) \leq |cd(H)|$ by induction argument (Here H is a proper subgroup of G, since q does not divide the order of H). Now we have a normal Hall subgroup H for which Taketa inequality holds and the factor group G/H is a q-group. So by Corollary 12.16 of [6] we have $dl(G) \leq |cd(G)|$. $\square$

As a preparation for the proof of the following theorems we prove the following proposition:

Proposition 2.7. *Let $\mathcal{P}$ be a class of finite solvable groups which is closed with respect to taking quotients. Suppose there exists a group in $\mathcal{P}$ for which the Taketa inequality is not true and let G be such a group of smallest possible order. Then the following hold:*

(i) $G^{(n-1)}$ is the unique minimal normal subgroup of G where $n = dl(G)$,

(ii) $cd(G/G^{(n-1)}) = cd(G)$,

(iii) $dl(G) = |cd(G)| + 1$,

(iv) $F(G)$, the Fitting subgroup of G, is a p-group for some prime p.

Furthermore if G'' is nilpotent, then

(v) p divides the index $|G : G'|$.

Proof. First assume that G has two distinct minimal normal subgroups M and N. Thus G is isomorphic to a subgroup of $G/M \times G/N$ since $M \cap N = 1$. As the Taketa inequality is true for both G/M and G/N we get $dl(G) \leq \max\{dl(G/M), dl(G/N)\} \leq \max\{|cd(G/M)|, |cd(G/N)|\} \leq |cd(G)|$. But this is a contradiction. So G has a unique minimal normal subgroup and consequently $F(G)$ is a p-group for some prime p. This completes the proof of (iv).

Let M be the unique minimal normal subgroup of G. In this case, M is abelian by the solvability of G and so $dl(G) \leq dl(M) + dl(G/M) = 1 + dl(G/M) \leq 1 + |cd(G/M)| \leq 1 + |cd(G)| \leq dl(G)$.

So we have $dl(G) = |cd(G)| + 1$, $|cd(G/M)| = |cd(G)|$, $dl(G/M) = dl(G) - 1 = n - 1$.

Since $G^{(n-1)}$ is non-trivial normal subgroup of G, M is contained in $G^{(n-1)}$. The equation $dl(G/M) = n - 1$ yields that $1 = \overline{G}^{(n-1)} = \overline{G^{(n-1)}}$ where $\overline{G} = G/M$. So we have $M = G^{(n-1)}$. This gives the proof of (i), (ii), (iii).

Now suppose that G'' is nilpotent. In this case, $G'' \subseteq F(G)$ and so G'' is a p-group. To prove (v), we will assume that p does not divide the index $|G : G'|$ and show that $dl(G) \leq |cd(G)|$ which is a contradiction. This will complete the proof. Since G'' is a p-group there exists a Sylow p-subgroup P of G' containing G''. It follows that P is normal in G. Since we assume that p does not divide the index $|G : G'|$, P is a normal Hall subgroup of G for which Taketa inequality holds. Clearly we may assume that $1 \neq P'$ since $G'' \subseteq P$ and Taketa inequality holds for groups $dl(G) \leq 3$. Since $1 \neq P'$, we have $dl(G/P') \leq |cd(G/P')|$. Finally we have $dl(G) \leq |cd(G)|$ by Proposition 2.1. □

Theorem 2.8. *Let G be a solvable group. Assume that for all $\chi, \psi \in Irr(G)$, $\ker\chi = \ker\psi$ if $1 < \chi(1) = \psi(1)$. Then $dl(G) \leq |cd(G)|$.*

Proof. Since $Irr(G/N) \subseteq Irr(G)$, the hypothesis is inherited by factor groups. Suppose the theorem is false and let G be a minimal counter example to this theorem. Then by Proposition 2.7, G has a unique minimal normal subgroup M and $cd(G/M) = cd(G)$.

Clearly we may assume $1 \neq G'$ so that $M \subseteq G'$ and $cd(G|M) \subseteq cd(G|G') = cd(G) - \{1\}$. Let $k \in cd(G|M)$. In this case, $1 \neq k$ and there exists an irreducible character χ of G such that $\chi(1) = k$ and $M \nsubseteq \ker\chi$. On the other hand, $k \in cd(G) = cd(G/M)$ and so there exists an irreducible character ψ of G such that $M \subseteq \ker\psi$ and $\psi(1) = k$. But by hypothesis, $\ker\chi = \ker\psi$ which is a contradiction. So we are done. □

Y. Berkovich, D. Chillag and M. Herzog have classified the finite groups in which the degrees of nonlinear irreducible characters are distinct and shown that such groups have at most three distinct irreducible character degrees [2]. So these groups satisfy the Taketa inequality. In the following corollary we have the same conclusion without exploring the structure of these groups.

Corollary 2.9. *Let G be a solvable group in which distinct nonlinear irreducible characters have distinct degrees. Then, $dl(G) \leq |cd(G)|$.*

Proof. This is an immediate corollary of Theorem 2.8. □

Let G be a finite group and $|G| = p_1^{\alpha_1} \dots p_r^{\alpha r}$ where $p_1, \dots, p_r$ are distinct primes and $\alpha_1, \dots, \alpha_r$ are non negative integers. We will denote the maximum of the α_i's by $\delta(G)$. Suppose that $\delta(G) \leq 2$. Then all Sylow subgroups of G are abelian and so $dl(G) \leq |cd(G)|$ as mentioned above. The next theorem gives a slightly better bound by putting an additional hypothesis:

Theorem 2.10. *Let G be a group and $k \in \{1, 2, 3, 4, 5\}$. If $G^{(k)}$ is nilpotent and $\delta(G) \leq 13 - 2k$ then $dl(G) \leq |cd(G)|$.*

Proof. Fix a $k \in \{1, 2, 3, 4, 5\}$ and suppose $G^{(k)}$ is nilpotent, $\delta(G) \leq 13 - 2k$. We will assume that the assertion is false and look for a contradiction. Let G be a minimal counter example to the assertion. In this case $6 \leq |cd(G)|$ by [4], [3] and [7]. Clearly the condition is inherited by factor groups and so we can apply Proposition 2.7. Then $n = dl(G) = |cd(G)| + 1 \geq 7$ and $F(G)$ is a p-group for some prime number p. By hypothesis $G^{(k)}$ is nilpotent and so $G^{(k)} \subseteq F(G)$. Thus $G^{(k)}$ is a p-group and so contained in a Sylow p-subgroup P of $G^{(k-1)}$ and P has to be normal in G. If $P' = 1$, then $G^{(k+1)} \subseteq P' = 1$ and so $7 \leq dl(G) \leq k + 1 \leq 6$ which is a contradiction. So P' is nontrivial so that $|G/P'| < |G|$ and hence $dl(G/P') \leq |cd(G/P')|$ by the minimality of G. Thus we see by Proposition 2.1 that P is not a Sylow p-subgroup of G.

When we consider the hypothesis $\delta(G) \leq 13 - 2k$ together with the last paragraph, we have that the order of P which is the p-part of the order of $G^{(k-1)}$ divides p^{12-2k} so that $cd(P) \subseteq \{1 = p^0, p, \dots, p^{5-k}\}$. Thus $n - k = dl(G^{(k)}) \leq dl(P) \leq |cd(P)| \leq 6 - k$ and so $n \leq 6$. But this is a contradiction since $7 \leq n$ by the first paragraph. □

Corollary 2.11. *Let G be a group. If G' is supersolvable and $\delta(G) \leq 9$ then $dl(G) \leq |cd(G)|$.*

Proof. This is an immediate consequences of Theorem 2.10 since the derived subgroup of a supersolvable group is nilpotent. □

Theorem 2.12. *Let G be a group with supersolvable derived subgroup. Suppose that G/G' is a p-group for some prime p and $2^k \not\equiv 1(p)$ for $k = 1, \dots, n$ where $|G|_2 = 2^n$. Then $dl(G) \leq |cd(G)|$.*

Proof. Let G be a minimal counter example to the Theorem. Since the conjecture is true for groups of odd order by [1], the order of G is even and $p \neq 2$ by Corollary 2.3. Let H be the unique $2'$-Hall subgroup of G' so that $H \lhd G$ and let $S \in Syl_p(G)$. Then, SH is a proper subgroup of G.If $SH \lhd G$ then $G/SH \cong G'/(G' \cap SH)$, but as G'' is a p-group by Proposition 2.7 we have $G'' \leq (G' \cap SH)$. Therefore G/SH is abelian which implies that $G' \leq SH$ and hence $G = SH$ which is not the case. So SH/H is a Sylow p-subgroup of G/H which is not normal. Thus we conclude that $1 < [G/H : N_{G/H}(SH/H)] \equiv 1\,(p)$. But $[G/H : N_{G/H}(SH/H)]$ divides $[G' : H]$ which is a power of 2. This contradiction completes the proof. □

Acknowledgements. We would like to thank to the referee for his/her suggestions which improved our original paper. The statements of Proposition 2.1 and Theorem 2.2 belong essentially to him/her.

The work of the first author was supported by Scientific Research Projects Coordination Unit of Istanbul University. The project number is 4383.

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