LETTER TO THE EDITOR

Convergence of threshold networks using their dissipative system model

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SUMMARY

A new dynamical energy system model representation is given for threshold networks. Inspired by the relation between stability and dissipativeness of dynamical systems, the convergence property of threshold networks is investigated. Using the energy function inherent within the given model a condition, namely the dissipativeness of the dynamical system, necessary and sufficient condition for the convergence of the threshold network to a fixed point, is given. Also, an easy to check inequality is stated to test the convergence of the threshold network. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: threshold networks; neural networks; dissipativeness; stability

INTRODUCTION

Results on convergence of threshold networks have found applications in various fields, to name some: binary stack filters, neural networks and cellular neural networks [1–4]. On the other hand, the relation between stability and dissipativeness of dynamical systems for linear-in control, partially linear and non-linear systems has been studied in continuous time [5–9]. Some results for discrete-time, linear-in control systems are also obtained [10,11]. The aim of this work is to establish a link between the results obtained for convergence properties of threshold networks and the results obtained for stability of dissipative dynamical systems. In the sequel, threshold networks will be shown to have a non-linear discrete-time dynamical system description with an energy function. Using this description some new results, (e.g. ‘equivalence of dissipativeness’ to...
‘convergence to a fixed point of the threshold network’), inequality (10) on convergence properties, will be obtained based on concepts derived for dissipative dynamical systems. An example will be considered to clarify the conditions obtained for the convergence of threshold networks.

DISSIPATIVE SYSTEMS

Given the following discrete dynamical system description:

\[ x(k + 1) = f(x(k), u(k)) \]  
\[ y(k) = g(x(k), u(k)) \]

with consumed energy function

\[ e(K, K_0, u, y) = \sum_{k=K_0}^{K-1} w(u(k), y(k)) \]

where \( w(u, y) \) is called supply rate in Reference [10], the system is dissipative if and only if for each \( x_0 \) of state space, \( E_A(x_0) < + \infty \), where

\[ E_A(x_0) = \sup_{K \geq 0} \left\{ - \sum_{k=0}^{K-1} w(u(k), y(k)) \right\} \]

with the supremum being taken over all \( K \geq 0 \) and all trajectories with initial state \( x_0 \).

Since, the use of internal energy function is quite common in investigating dissipativeness [8], its definition is given below.

A function \( E_I \) mapping state space into non-negative real numbers and satisfying

\[ E_I(x(K_2)) - E_I(x(K_1)) \leq \sum_{k=K_1}^{K_2-1} w(u(k), y(k)) \]

for all input-state trajectories \( \{u(k), y(k)\} \) on \([K_1, K_2]\) and all \( 0 \leq K_1 \leq K_2 \) is called internal energy function. The system given by (1)–(3) is dissipative if and only if there exists an internal energy function [11]. The concepts given in this section will be used in the following section to obtain some results on the convergence properties of threshold networks.

THRESHOLD NETWORKS

Threshold networks are defined by the following equalities:

\[ x(k + 1) = S(Ax(k) + b) \]
\[ y(k) = Ax(k) + b \]
Here the $i$th row of $S(\mathbf{A}x + b) \in \mathbb{R}^n$, $S_i$ is a threshold function performing the non-linear mapping as defined in the following:

$$S_i(\mathbf{a}_i x + b_i) = \begin{cases} 
1 & \text{for } \mathbf{a}_i x + b_i \geq 0 \\
-1 & \text{for } \mathbf{a}_i x + b_i \leq 0 
\end{cases}$$

where $\mathbf{a}_i$ and $b_i$ denote the $i$th row of $\mathbf{A}$ and $\mathbf{b}$, respectively. This structure is called ‘general linear threshold network structure’ since $S(\cdot)$ is defined through a linear inequality. Matrix $\mathbf{A} \in \mathbb{R}^n \times \mathbb{R}^n$ denotes any arbitrary matrix and $\mathbf{b} \in \mathbb{R}^n$ is the constant bias (threshold vector) and $x(k) \in \{-1, 1\}^n$.

An important problem is the convergence of the network to a fixed point. Since the state space is finite, it is well known that this network either converges (given an initial state the trajectory converges to a fixed point at most in $2^n$ steps) or oscillates (given an initial state the trajectory converges to a limit cycle in finite time). In order to obtain some results for the convergence of the network to a fixed point, dissipativeness of the network will be investigated by defining the supply rate to be

$$w(\mathbf{u}, \mathbf{y}) = -z (\mathbf{A}S(\mathbf{y}) + \mathbf{b} - \mathbf{y})^T (\mathbf{A}S(\mathbf{y}) + \mathbf{b} - \mathbf{y})$$

(8)

with $z > 0$.

Given a threshold network by (6) and (7) and the consumed energy function (3) related to $w$ as defined by (8) it is obvious that, if the network converges to a fixed point for each initial state then it is dissipative as terms in the summation in (4) will be zero after a finite number of terms. The opposite is also true as stated in the following proposition.

**Proposition.** A threshold network given by (6) and (7) and its consumed energy function by (3), and (8) converges to a fixed point for each initial state, if and only if the network is dissipative.

The ‘only if’ part being shown above, to show ‘if’ part, using the definition given by (4), the following inequality can be written along trajectories:

$$E_{\mathbf{A}}(\mathbf{x}_0) \geq \sum_{k=0}^{K-1} z \| \mathbf{A}(\mathbf{x}(k + 1) - \mathbf{x}(k)) \|^2 \quad \forall \mathbf{x}_0 \in \Sigma, \quad \forall K \in J^+$$

(9)

Let $v(k) = x(k+1) - x(k)$; $v(k) \in \{-2, 0, 2\}^n$. If $\mathbf{A}v(k) \neq 0$ in every finite interval then there must exist a $K_0$ (i.e. after the transient dies) and $k_1 \geq K_0$ such that $v(k_1) \neq 0$, $\mathbf{A}v(k_1 + n\bar{K}) = \mathbf{A}v(k_1)$ for all $n \in J^+$ where $\bar{K}$ is the period of the limit cycle (which the threshold network must enter). Hence, the summation in (9) can not be finite contradicting dissipativeness. Thus $\mathbf{A}v(k) = 0$ for all $k$ after some finite interval $[0, \bar{K}]$ and if $\mathbf{A}v(k) = 0$ implies $v(k) = 0$ after the same interval then the proposition is satisfied. If not, it will be shown that no limit cycle can be formed outside $[0, \bar{K}]$.

Let $x(\bar{K} + j)$ form a limit cycle for $j = 1, 2, \ldots, k_0$, then $\mathbf{A}(x(\bar{K} + j + 1) - x(\bar{K} + j)) = \mathbf{A}v(\bar{K} + j) = 0$ for $j = 1, 2, \ldots, k_0 - 1$ which implies $\mathbf{A}x(\bar{K} + j + 1) = \mathbf{A}x(\bar{K} + j)$ for $j = 1, 2, \ldots, k_0 - 1$ giving in turn $S(\mathbf{A}x(\bar{K} + j + 1) + b) = S(\mathbf{A}x(\bar{K} + j))$, i.e. a limit cycle reduces to a fixed point proving the proposition.

The above proposition gives a qualitative result which shows the equivalence between dissipativeness and convergence to a fixed point of the threshold network. Inequality (10), given in terms of threshold network parameters, can be used as a testing criterion for...
dissipativeness.

\[
[S^2_{A,b}(x) - S_{A,b}(x)]^T[S^2_{A,b}(x) - S_{A,b}(x)] \leq [S_{A,b}(x) - x]^T(I - \alpha A^T)A[S_{A,b}(x) - x]
\]  

(10)

where \(S_{A,b}(x) = S(Ax + b)\) and \(S^2_{A,b}(x) = S(AS_{A,b}(x) + b)\) with \(0 < \alpha < 1\), \(\forall x \in \Sigma\). Now defining \(E_t(x) = [S_{A,b}(x) - x]^T[S_{A,b}(x) - x]\), writing inequality (10) along trajectories and summing up both sides from \(k = 0\) to \(K - 1\) the following inequality can be written:

\[
E_t(x(K)) - E_t(x(0)) \leq - \sum_{k=0}^{K-1} (x(k+1) - x(k))^T \alpha A^T A(x(k+1) - x(k))
\]

which shows that \(E_t(\cdot) : \Sigma \to R^+\) is an internal energy function. So the system is dissipative.

As an example consider the threshold network with

\[
A = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

It is easy to show that trajectories starting with initial conditions \([1 \ 1]^T\) and \([-1 \ -1]^T\) converge to a fixed point \([1 \ 1]^T\), while the trajectories starting with other initial conditions enter a limit cycle. This system is not dissipative since for initial conditions \([1 \ -1]^T\) and \([-1 \ 1]^T\) it is not possible to obtain a finite value for (4). As expected it is also not possible to find \(0 < \alpha < 1\) satisfying inequality (10).

CONCLUSION

In this work the threshold network is considered as a dynamical system for which an an energy function is defined. Necessary and sufficient condition for the convergence properties of threshold networks is given using the relation between convergence and dissipativeness. Also an inequality, i.e. (10), is given to check the dissipativeness of the threshold network. These results are obtained using concepts related with dissipative dynamical systems.

REFERENCES


