ISSN 0001-4346, Mathematical Notes, 2007, Vol. 81, No. 4, pp. 440–448. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © O. A. Veliev, 2007, published in Matematicheskie Zametki, 2007, Vol. 81, No. 4, pp. 496–506.

Non-Self-Adjoint Sturm–Liouville Operators with Matrix Potentials

O. A. Veliev*

Dogus University

Received August 5, 2005

Abstract—We obtain asymptotic formulas for non-self-adjoint operators generated by the Sturm–Liouville system and quasiperiodic boundary conditions. Using these asymptotic formulas, we obtain conditions on the potential for which the system of root vectors of the operator under consideration forms a Riesz basis.

DOI: 10.1134/S0001434607030200

Key words: *Sturm–Liouville operator, non-self-adjoint operator, quasiperiodic boundary condition, Riesz basis, root function, Jordan chain, Bessel operator.*

In this paper, we study the differential operator $L_t(Q)$ generated by the differential expression

$$l(y) = -y''(x) + Q(x)y(x)$$

in the space $L_2^m[0,1]$ and by the quasiperiodic boundary conditions

$$y'(1) = e^{it}y'(0), \qquad y(1) = e^{it}y(0), \tag{1}$$

where $t \in [0, 2\pi)$.

Here $L_2^m[0,1]$ is the space of vector functions $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$ with coordinates $f_k(x) \in L_2[0,1]$, $k = 1, 2, \dots, m$, and $Q(x) = \{b_{i,j}(x)\}$ is a matrix of dimensions $m \times m$ whose elements are complex-valued integrable functions $b_{i,j}(x)$. The norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$ on the space $L_2^m[0,1]$ are defined by

$$\|f\| = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}, \qquad (f,g) = \int_0^1 \langle f(x), g(x)\rangle\,dx,$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ is the norm and the inner product on $\mathbb{C}^m$.

Such boundary-value problems play a fundamental role in the spectral theory of the differential operator L generated in the space $L_2^m(-\infty,\infty)$ by a differential expression $l(y)$ with periodic potentials $Q(x)$. This can be explained by the fact that the spectrum of the operator L is the union of the spectra of the operators L_t when t ranges over the closed interval $[0, 2\pi)$ (see [1]).

Let us give an outline of the paper. It is readily seen from the classical results (see [2]) that the eigenvalues of the operator $L_t(Q)$ consist of m sequences of numbers

$$\{\lambda_{k,1} : k \in \mathbb{Z}\}, \qquad \{\lambda_{k,2} : k \in \mathbb{Z}\}, \qquad \dots, \qquad \{\lambda_{k,m} : k \in \mathbb{Z}\} \tag{2}$$

that lie inside circles of radii $O(|k|^{1-1/m})$ whose centers coincide with the eigenvalues $(2k\pi + t)^2$ of the operator $L_t(0)$ (see Theorem 1). Further, we prove that the eigenvalues $\lambda_{k,j}$ of the operator $L_t(Q)$ lie inside $O(\ln|k|/k)$ neighborhoods whose centers are the eigenvalues of the operator $L_t(C)$ with a constant potential $C = \int_0^1 Q(x)\,dx$. To do this, we express the operator $L_t(Q)$ as a perturbation $L_t(C)$ by the operator $Q(x) - C$. Therefore, we first analyze the eigenvalues and eigenfunctions of the operator $L_t(C)$, and, after that, we obtain formulas (17)–(19) relating the eigenvalues and the eigenfunctions of

*E-mail: oveliev@dogus.edu.tr

the operators $L_t(Q)$ and $L_t(C)$. Using these formulas and Lemmas 1–3, we find the asymptotics of the eigenvalues and eigenfunctions of the perturbed operator $L_t(Q)$. These results are stated in Theorem 2; they constitute the main contents of the paper.

Note that, in order to obtain asymptotic formulas with a remainder of the form $O(k^{-1})$ for the eigenvalues $\lambda_{k,j}$ of the operator $L_t(Q)$, the classical method of asymptotic expansions of the solutions of the matrix equation $-Y'' + Q(x)Y = \lambda Y$ is usually used, and is assumed that the elements of the matrix $Q(x)$ are differentiable (see [2]–[5]). The method proposed here allows us to obtain asymptotic formulas for the eigenvalues $\lambda_{k,j}$ and the normalized eigenfunctions $\Psi_{k,j}(x)$ of the operator $L_t(Q)$ with remainder $O(k^{-1}\ln|k|)$ when the elements $b_{i,j}(x)$ of the matrix $Q(x)$ belong to the space $L_1[0,1]$. As a consequence of the resulting asymptotic formulas, by using Bari's theorem, we find that, for $t \neq 0, \pi$, the root functions of the operator $L_t(Q)$ form a Riesz basis whenever all the eigenvalues of the matrix potential C are simple.

It should be noted here that general results concerning the Riesz basis property of ordinary differential operators of higher order and of more complicated boundary-value problems when the equations and the boundary conditions contain nonlinear functions of the spectral parameter were obtained *in the scalar case* by Shkalikov [6]–[8] (the history of the problem and the corresponding references are given in [7]). He also noted [9] that methods from [6] and [7] can be extended to cover the *matrix case*, essentially, without any changes. A generalization of the results from [8] to the matrix case was carried out by Luzhina [10]. Shkalikov's theorem [6] [9] is stated as follows: *The root functions of an ordinary differential operator with matrix integrable coefficients which is generated by regular boundary conditions form a Riesz basis with brackets comprising only functions corresponding to eigenvalues approaching each other*. In our case, we show that only the prime roots of the matrix C guarantee the **ordinary** Riesz property without brackets.

Theorem 1. *The boundary conditions* (1) *are regular for all* $t \in [0, 2\pi)$. *For* $t \neq 0, \pi$, *the eigenvalues of the operator* $L_t(Q)$ *form* m *sequences* (2) *having the identical asymptotics*

$$\lambda_{k,j}(t) = (2\pi k + t)^2 + O(k^{1-1/m}), \qquad k = \pm N, \pm(N+1), \dots, \quad j = 1, 2, \dots, m. \tag{3}$$

Here we assume that the number N *is sufficiently large.*

Proof. By the definition from [2], the boundary conditions are said to be *regular* if the numbers θ_{-m}, θ_m defined by the equality

$$\theta_{-m}s^{-m} + \theta_{-m+1}s^{m-1} + \dots + \theta_m s^m = \det M(m)$$

are nonzero. Here

$$M(m) = \begin{bmatrix} (e^{it} - s)iI, & \left(e^{it} - \dfrac{1}{s}\right)(-i)I \\ (e^{it} - s)I, & \left(e^{it} - \dfrac{1}{s}\right)I \end{bmatrix},$$

and I is the unit matrix of dimension $m \times m$. Let us prove the equality

$$\det M(m) = \left(-2ie^{it}s + 2i + 2ie^{2it} - 2ie^{it}\frac{1}{s}\right)^m, \tag{4}$$

which implies regularity, since, in this case, the two numbers $\theta_m = (-2ie^{it})^m$, $\theta_{-m} = (-2ie^{it})^{-m}$ are not equal to zero.

Note that, for $m = 1$, formula (4) is valid, because

$$\det M(1) = \begin{vmatrix} (e^{it} - s)i & \left(e^{it} - \dfrac{1}{s}\right)(-i) \\ (e^{it} - s) & \left(e^{it} - \dfrac{1}{s}\right) \end{vmatrix} = -2ie^{it}s + 2i + 2ie^{2it} - 2ie^{it}\frac{1}{s}.$$

Using the obvious equality

$$\det M(m) = \begin{vmatrix} M_1 & 0 \\ 0 & M_{p-1} \end{vmatrix},$$

we obtain $\det M(m) = (\det M(1)) \det M(m-1)$. This relation allows us to conclude the proof of formula (4) by induction. Thus, regularity is proved.

It follows from formula (4) that, for $t \neq 0, \pi$, the numbers $s_1 = e^{it}$,, $s_2 = e^{-it}$ are distinct and both are roots of multiplicity m of the equation

$$\det M(m) = 0.$$

But, in that case, it follows from [2, Chap. 3, Theorem 2] that to each of the roots s_1 and s_2 there corresponds m sequences of eigenvalues of the operator $L_t(Q)$ possessing, up to $O(k^{1-1/m})$, the identical asymptotics

$$\lambda_{k,j}^{(1)} = (2k\pi + t)^2 + O(k^{1-1/m}), \quad \lambda_{k,j}^{(2)} = (2k\pi - t)^2 + O(k^{1-1/m}), \qquad k = N, N+1, \dots,$$

where $j = 1, 2, \dots, m$. Obviously, both these sequences can be written in the form (3). The theorem is proved. □

In the subsequent estimates, by $C_1, C_2, \dots$, we denote different positive constants whose values are inessential. Formula (3) implies the following: the eigenvalues $\lambda_{k,j}$ of the operator $L_t(Q)$ are close to the eigenvalues $(2k\pi + t)^2$ of the operator $L_t(0)$ with zero potential. Besides, for $t \neq 0, \pi$, the eigenvalues $\lambda_{k,j}$ are far away from the numbers $(2n\pi + t)^2$ for $n \neq k$. More exactly, for $t \neq 0, \pi$ the following inequalities hold:

$$|\lambda_{k,j} - (2k\pi + t)^2| < C_1 |k|^{1-1/m}, \qquad |\lambda_{k,j} - (2(-k)\pi + t)^2| > C_2 |k|,$$
$$|\lambda_{k,j} - (2\pi n + t)^2| > C_2 \big(\big||k| - |n|\big|\big)(|k| + |n|) \qquad \text{if} \quad n \neq \pm k. \tag{5}$$

These inequalities imply the following relations:

$$\sum_{n>d} \frac{1}{|\lambda_{k,j} - (2\pi n + t)^2|} < \frac{C_3}{d} \qquad \text{for all} \quad d > 2|k|, \tag{6}$$

$$\sum_{n \neq k} \frac{1}{|\lambda_{k,j} - (2\pi n + t)^2|} = O\left(\frac{\ln |k|}{k}\right), \tag{7}$$

$$\sum_{n \neq k} \frac{1}{|\lambda_{k,j} - (2\pi n + t)^2|^2} = O\left(\frac{1}{k^2}\right). \tag{8}$$

Here the summation is taken over the values of the index n, $t \neq 0, \pi$, and the number $k \geq N$ is assumed sufficiently large.

Note that

$$\varphi_{n,1} = \begin{pmatrix} e^{i(2\pi n + t)x} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \varphi_{n,2} = \begin{pmatrix} 0 \\ e^{i(2\pi n + t)x} \\ \vdots \\ 0 \end{pmatrix}, \qquad \dots, \qquad \varphi_{n,m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{i(2\pi n + t)x} \end{pmatrix}$$

are the eigenfunctions of the operator $L_t(0)$ corresponding to the eigenvalues $(2\pi n + t)^2$. The multiplicity of the eigenvalue $(2\pi n + t)^2$ is equal to m, and the corresponding proper subspace coincides with $E_n = \operatorname{span}\{\varphi_{n,1}, \varphi_{n,2}, \dots, \varphi_{n,m}\}$. The adjoint operator to $L_t(Q)$ coincides with $L_t(Q^*)$, where $Q^*(x)$ is the conjugate matrix to $Q(x)$. Since the boundary condition (1) is self-adjoint, it follows that the operator $L_t(Q)$ is self-adjoint if $Q(x)$ is a symmetric matrix. Therefore, $L_t^*(0) = L_t(0)$.

Now let us find the eigenvalues and root functions of the operator $L_t(C)$. We introduce the following notation. Suppose that the matrix C has p distinct eigenvalues $\mu_1, \mu_2, \dots, \mu_p$ with multiplicities $m_1, m_2, \dots, m_p$, respectively. Then $m_1 + m_2 + \dots + m_p = m$. Denote by $u_{j,1}, u_{j,2}, \dots, u_{j,s_j}$ the eigenfunctions corresponding to the eigenvalues μ_j and by $u_{j,s,l}$, $l = 1, \dots, r_s - 1$, the associated functions corresponding to the eigenfunctions $u_{j,s}$. Denote by $r_{j,s}$ the lengths of the corresponding Jordan chains (the dimensions of the Jordan blocks), and let $r_{j,1} + r_{j,2} + \dots + r_{j,s_j} = m_j$. The number

$$r_j = \max_s r_{j,s} \tag{9}$$

defines the maximal length of the Jordan chain corresponding to the eigenvalue μ_j. It is easy to see that, for $s = 1, 2, \dots$, the functions $u_{j,s} e^{i(2\pi k+t)x}$ are the eigenfunctions of the operator $L(C)$ corresponding to the eigenvalue $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$, while $u_{j,s,r} e^{i(2\pi k+t)x}$, where $r = 1, 2, \dots$, are the corresponding associated functions.

In the subsequent relations, it is convenient to number the eigenvalues of the matrix C, counting multiplicity $\mu_1, \mu_2, \dots, \mu_m$. The normalized eigenfunctions corresponding to the eigenvalue μ_j are denoted by v_j. The associated functions corresponding to the vector v_j are denoted by $v_{j,1}, v_{j,2}, \dots$. In our notation, the eigenvalues, the eigenfunctions, and the associated functions of the operator $L_t(C)$ can be written as

$$\mu_{k,j} = (2\pi k + t)^2 + \mu_j, \qquad \Phi_{k,j}(x) = v_j e^{i(2\pi k+t)x}, \qquad \Phi_{k,j,s}(x) = v_{j,s} e^{i(2\pi k+t)x}.$$

Similarly, the eigenvalues, the eigenfunctions, and the associated functions of the operator $L_t^*(C)$ take the form

$$\overline{\mu_{k,j}}, \qquad \Phi^*_{k,j}(x) = v^*_j e^{i(2\pi k+t)x}, \qquad \Phi^*_{k,j,s}(x) = v^*_{j,s} e^{i(2\pi k+t)x}.$$

Here v^*_j and $v^*_{j,s}$ are the eigen and associated vectors of the matrix C^* corresponding to $\overline{\mu_j}$. By definition, we have

$$(L^*(C) - \overline{\mu_{k,j}})\Phi^*_{k,j}(x) = 0, \tag{10}$$

$$(L^*(C) - \overline{\mu_{k,j}})\Phi^*_{k,j,s}(x) = \Phi^*_{k,j,s-1}(x), \tag{11}$$

where $\Phi^*_{k,j,0}(x) = \Phi^*_{k,j}(x)$. Let us multiply the equality

$$L(Q)\Psi_{k,j}(x) = \lambda_{k,j}\Psi_{k,j}(x) \tag{12}$$

scalarly by $\Phi^*_{k,j}(x)$. Using the equality $L(Q) = L(C) + (Q - C)$ and (10), we obtain

$$(\lambda_{k,j} - \mu_{k,j})(\Psi_{k,j}(x), \Phi^*_{k,j}(x)) = ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)). \tag{13}$$

Let us multiply (12) scalarly by $\Phi^*_{k,j,1}(x)$ and use (11). Then we obtain

$$(\lambda_{k,j} - \mu_{k,j})(\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)) = (\Psi_{k,j}(x), \Phi^*_{k,j}(x)) + ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)).$$

Multiplying this equality by $(\lambda_{k,j} - \mu_{k,j})$ and using relation (13), we can write

$$\begin{aligned}(\lambda_{k,j} - \mu_{k,j})^2(\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)) &= ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j}(x)) \\ &\quad + (\lambda_{k,j} - \mu_{k,j})((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j,1}(x)).\end{aligned} \tag{14}$$

In view of relations (11)–(14), we find similarly

$$(\lambda_{k,j} - \mu_{k,j})^{s+1}(\Psi_{k,j}(x), \Phi^*_{k,j,s}(x)) = \sum_{p=0}^{s} (\lambda_{k,j} - \mu_{k,j})^p ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j,p}(x)). \tag{15}$$

To estimate the expressions $((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,j,p}(x))$, $(\Psi_{k,j}(x), \Phi^*_{k,j,s}(x))$ in formula (15), we use (6)–(8) and the following formulas:

$$(\lambda_{k,j} - (2\pi n + t)^2)(\Psi_{k,j}(x), \varphi_{n,s}(x)) = (\Psi_{k,j}(x), Q^*(x)\varphi_{n,s}(x)), \tag{16}$$

$$(\Psi_{k,j}(x), \varphi_{n,s}(x)) = \frac{(\Psi_{k,j}(x), Q^*(x)\varphi_{n,s}(x))}{\lambda_{k,j} - (2\pi n + t)^2} \qquad \forall\, n \neq k. \tag{17}$$

Formula (17) follows from (16) and (5), while formula (16) is obtained after the scalar multiplication of relation (12) by $\varphi_{n,s}(x)$ if we take the following relations into account:

$$L_t(0)\varphi_{n,s}(x) = (2\pi n + t)^2\varphi_{n,s}(x), \qquad L_t^*(0) = L_t(0).$$

Lemma 1. *For $t \neq 0, \pi$, we have*

$$(\Psi_{k,j}(x), Q^*(x)\varphi_{n,s}(x)) = \sum_{q=1,2,\dots,m}\sum_{p\in\mathbb{Z}} b_{s,q,n-p}(\Psi_{k,j}(x), \varphi_{p,q}(x)), \tag{18}$$

$$|(\Psi_{k,j}(x), Q^*(x)\varphi_{n,s}(x))| < C_4. \tag{19}$$

Here $n \in \mathbb{Z}$, $|k| \geq N$, $s, j = 1, 2, \dots, m$ and

$$b_{s,q,n-p} = \int_0^1 b_{s,q}(x)e^{2\pi i(p-n)x}\,dx.$$

Proof. It follows from the inclusion $Q(x)\Psi_{k,j}(x) \in L_1^m[0,1]$ that

$$\lim_{n\to\infty}(Q(x)\Psi_{k,j}(x), \varphi_{n,s}(x)) = 0 \qquad \text{for} \quad s = 1, 2, \dots, m.$$

Therefore, there exists a positive constant $C(k,j)$ and indices n_0, j_0 such that

$$\max_{\substack{n\in\mathbb{Z}\\ s=1,2,\dots,m}} |(\Psi_{k,j}, Q^*(x)\varphi_{n,s})| = |(\Psi_{k,j}, Q^*(x)\varphi_{n_0,j_0})| = C(k,j). \tag{20}$$

Using (20) and (17), and, after that, (6), we obtain

$$|(\Psi_{k,j}(x), \varphi_{n,s}(x))| \leq \frac{C(k,j)}{|\lambda_{k,j} - (2\pi n + t)^2|}, \tag{21}$$

$$\sum_{n:n>d} |(\Psi_{k,j}(x), \varphi_{n,s}(x))| < \frac{C_3 C(k,l)}{d} \qquad \text{if} \quad d > 2|k|.$$

Then the expansion of the function $\Psi_{k,j}(x)$ in the orthonormal basis $\{\varphi_{n,s}(x) : n \in \mathbb{Z},\ s = 1, 2, \dots, m\}$ is of the form

$$\Psi_{k,j}(x) = \sum_{\substack{p:|p|\leq d\\ q=1,2,\dots,m}} (\Psi_{k,j}(x), \varphi_{p,q}(x))\varphi_{p,q}(x) + g_d(x), \tag{22}$$

where

$$\sup_{x\in[0,1]} |g_d(x)| < \frac{C_3 C(k,l)}{d}.$$

Multiplying scalarly relation (22) by the function $Q^*(x)\varphi_{n,s}(x)$ and letting $d \to \infty$, we obtain (18).

Now let us prove inequality (19). Repeating the procedure just carried out, letting $d \to \infty$, and extracting the terms with multipliers $(\Psi_{k,j}(x), \varphi_{k,i}(x))$ for $i = 1, 2, \dots, m$, we find

$$(\Psi_{k,j}(x), Q^*(x)\varphi_{n_0,j_0}(x)) = \sum_{i=1,2,\dots,m} b_{j_0,i,n_0-k}(\Psi_{k,j}, \varphi_{k,i}) + \sum_{\substack{n:n\neq k\\ i=1,2,\dots,m}} b_{j_0,i,n_0-n}(\Psi_{k,j}, \varphi_{n,i}). \tag{23}$$

Since

$$|b_{j,i,s}| \leq \max_{p,q=1,2,\dots,m}\int_0^1 |b_{p,q}(x)|\,dx < C_5 \qquad \forall\, j, i, s, \tag{24}$$

from (21) and (7) we obtain

$$\sum_{\substack{n:n\neq k\\ i=1,2,\dots,m}} b_{j_0,i,n_0-n}(\Psi_{k,j}, \varphi_{n,i}) = O\left(\frac{\ln|k|}{k}\right)C(k,j). \tag{25}$$

It follows from (24) that the first summand on the right-hand side of (23) can be estimated by mC_5. From relations (20), (23) and (25) we then obtain $|C(k,l)| < 2mC_5$, which implies estimate (19). □

Lemma 2. *For $t \neq 0, \pi$, we have*

$$(\Psi_{k,j}(x), (Q^*(x) - C^*)\Phi^*_{k,i,p}(x)) = O\biggl(\frac{\ln|k|}{k}\biggr), \tag{26}$$

where $i = 1, 2, \dots, m$ and $p = 0, 1, 2, \dots$.

Proof. By the equalities

$$\Phi^*_{k,n,p}(x) = v^*_{n,p} e^{i(2\pi k+t)x}$$

it suffices to prove that

$$(\Psi_{k,j}(x), (Q^*(x) - C^*)\varphi_{k,s}(x)) = O\biggl(\frac{\ln|k|}{k}\biggr) \tag{27}$$

for $s = 1, 2, \dots, m$. Let us use the obvious relation

$$(\Psi_{k,j}(x), C^*\varphi_{k,s}(x)) = \sum_{i=1,2,\dots,m} b_{s,i,0}(\Psi_{k,j}(x), \varphi_{k,i}(x))$$

and (18). Then

$$(\Psi_{k,j}, (Q^*(x) - C^*)\varphi_{k,s}(x)) = \sum_{\substack{n:n\neq k\\ i=1,2,\dots,m}} b_{s,i,k-n}(\Psi_{k,j}(x), \varphi_{n,i}(x)). \tag{28}$$

On the other hand, from (17) and (19) we obtain

$$|(\Psi_{k,j}(x), \varphi_{n,i}(x))| \le \frac{C_4}{|\lambda_{k,j} - (2\pi n + t)^2|} \tag{29}$$

for $n \neq k$ and $i = 1, 2, \dots, m$. Therefore, it follows from (24) and (7) that the right-hand side of (28) can be estimated by $O(k^{-1}\ln|k|)$. This yields the proof of estimate (27). □

Lemma 3. *For each eigenfunction $\Psi_{k,j}(x)$ of the operator $L_t(Q)$ for $|k| \ge N$, there exists a root function $\Phi^*_{k,i,s}(x)$ of the operator $L_t(C^*)$ for which the following condition is satisfied:*

$$|(\Psi_{k,j}(x), \Phi^*_{k,i,s}(x))| > C_6.$$

Proof. Since the system $\{\varphi_{n,i}(x) : i = 1, 2, \dots, m,\ n \in \mathbb{Z}\}$ is an orthonormal basis in $L_2^m[0,1]$, we have

$$\Psi_{k,j}(x) = \sum_{i=1,2,\dots,m} \biggl(\sum_{n\in\mathbb{Z}} (\Psi_{k,j}(x), \varphi_{n,i}(x))\varphi_{n,i}(x)\biggr),$$

$$\|\Psi_{k,j}\|^2 = \sum_{i=1,2,\dots,m} |(\Psi_{k,j}, \varphi_{k,i})|^2 + \sum_{i=1,2,\dots,m} \biggl(\sum_{n:n\neq k} |(\Psi_{k,j}, \varphi_{n,i})|^2\biggr). \tag{30}$$

On the other hand, it follows from (29) and (8) that

$$\sum_{i=1,2,\dots,m} \biggl(\sum_{n:n\neq k} |(\Psi_{k,j}, \varphi_{n,i})|^2\biggr) = O\biggl(\frac{1}{k^2}\biggr). \tag{31}$$

From the equalities $\|\Psi_{k,j}(x)\| = 1$ and (30) we obtain

$$\sum_{i=1,2,\dots,m} |(\Psi_{k,j}(x), \varphi_{k,i}(x))|^2 = 1 + O\biggl(\frac{1}{k^2}\biggr). \tag{32}$$

Therefore, the norm of the projection of the function $\Psi_{k,j}(x)$ on the subspace

$$E_k = \operatorname{span}\{\varphi_{k,1}, \varphi_{k,2}, \dots, \varphi_{k,m}\}$$

is equal to $1 + O(1/k^2)$. Using the equalities

$$\Phi^*_{k,j}(x) = v^*_j e^{i(2\pi k+t)x}, \qquad \Phi^*_{k,j,s}(x) = v^*_{j,s} e^{i(2\pi k+t)x}$$

and taking into account the fact that the vectors v_j, $v^*_{j,s}$, where $j = 1, 2, \dots$ and $s = 1, 2, \dots$, form a basis in $\mathbb{C}^m$, we obtain the proof of the lemma. □

Theorem 2. *For $t \neq 0, \pi$, the following assertions hold:*

(a) *All sufficiently large (in absolute value) eigenvalues of the operator $L_t(Q)$ lie inside the circles of radius $O((\ln|k|/k)^{1/r_j})$ centered at the points $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$ which are the eigenvalues the operator $L_t(C)$. Here $k \in \mathbb{Z}$, $j = 1, 2, \dots, m$, and the numbers r_j are defined in* (9)*; moreover, the number r_j is equal to 1 if the corresponding eigenvalue μ_j of the matrix C is semisimple, i.e., there are no corresponding Jordan blocks.*

(b) *Suppose that μ_j is a simple eigenvalue of the matrix C and $\lambda_{k,j}$ is an eigenvalue of the operator $L_t(Q)$ lying inside the circle of radius $a_j/2$ centered at the point $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$, where $a_j = \min_{i \neq j} |\mu_j - \mu_i|$. Then $\lambda_{k,j}$ is a simple eigenvalue of the operator $L_t(Q)$. Moreover, the numbers $\lambda_{k,j}$ and the corresponding eigenfunctions $\Psi_{k,j}(x)$ possess the asymptotics*

$$\lambda_{k,j}(t) = (2\pi k + t)^2 + \mu_j + O\left(\frac{\ln|k|}{k}\right), \tag{33}$$

$$\Psi_{k,j}(x) = v_j e^{i(2\pi k+t)x} + O\left(\frac{\ln|k|}{k}\right), \tag{34}$$

where v_j is the eigenvector of the matrix C corresponding to μ_j.

(c) *Suppose that all the eigenvalues $\mu_1, \mu_2, \dots, \mu_m$ of the matrix C are simple. Then there exists a number N such that all the eigenvalues $\lambda_{k,1}, \lambda_{k,2}, \dots, \lambda_{k,m}$ of the operator $L_t(Q)$ for $|k| \geq N$ are simple and satisfy the asymptotic formula* (33)*. The corresponding eigenfunctions $\Psi_{k,j}(x)$ have the asymptotics* (34)*. The system of all root functions of the operator $L_t(Q)$ forms a Riesz basis in the space $L_2^m(0,1)$.*

Proof. Note that, for $n = 0, 1, \dots, s$, formula (15) is also valid (with the index j of the numbers $\mu_{k,j}$ and functions $\Phi^*_{k,j,n}$ replaced by the index i):

$$(\lambda_{k,j} - \mu_{k,i})^{s+1} (\Psi_{k,j}(x), \Phi^*_{k,i,s}(x)) = \sum_{p=0}^{s} (\lambda_{k,j} - \mu_{k,i})^p ((Q(x) - C)\Psi_{k,j}(x), \Phi^*_{k,i,p}(x)). \tag{35}$$

Using this formula, estimate (26), and Lemma 3, we obtain

$$(\lambda_{k,j} - \mu_{k,i})^{s+1} = \sum_{p=0}^{s} (\lambda_{k,j} - \mu_{k,i})^p O\left(\frac{\ln|k|}{k}\right),$$

where $s + 1 \leq r_i$, and the number r_i is defined in (9). This proves assertion (a).

Let us prove assertion (b). For $i \neq j$, we have $|\lambda_{k,j} - \mu_{k,i}| > a_j/2$; therefore, from (26) and (35), we obtain

$$(\Psi_{k,j}(x), \Phi^*_{k,i,p}(x)) = O\left(\frac{\ln|k|}{k}\right), \qquad p = 1, 2, \dots .$$

This estimate and (31) prove the asymptotics of (34) for the normalized eigenfunctions corresponding to the eigenvalues $\lambda_{k,j}$. Here we take into account the fact that

$$\operatorname{span}\{\varphi_{k,1}, \varphi_{k,2}, \dots, \varphi_{k,m}\} = \operatorname{span}\{\Phi^*_{k,i,p}(x) : i = 1, 2, \dots,\ p = 0, 1, \dots\},$$

and that μ_j is a simple eigenvalue the matrix C.

Now let us prove that $\lambda_{k,j}$ is a simple eigenvalue of the operator $L(Q)$. If the geometric multiplicity of the eigenvalue $\lambda_{k,j}$ is greater than one, then there exist two corresponding mutually orthogonal eigenfunctions satisfying relation (34), but this cannot be true. Now, if there is a function $\Psi_{k,j,1}(x)$ which is an associated function for the eigenfunction $\Psi_{k,j}(x)$, then

$$(L(Q) - \lambda_{k,j})\Psi_{k,j,1}(x) = \Psi_{k,j}(x).$$

Let us multiply this equality scalarly by $\Psi^*_{k,j}(x)$, where $\Psi^*_{k,j}(x)$ is the eigenfunction of the adjoint operator $L^*(Q)$ corresponding to the eigenvalue $\overline{\lambda_{k,j}}$. Then we obtain

$$(\Psi_{k,j}, \Psi^*_{k,j}) = (\Psi_{k,i,1}(x), (L^*(Q) - \overline{\lambda_{k,j}}))\Psi^*_{k,j}(x) = 0. \tag{36}$$

The eigenfunctions of the adjoint operator $L^*(Q) = L(Q^*)$ also satisfy assertion (b) of Theorem 2 now written in the form

$$\Psi^*_{k,j}(x) = v^*_j e^{i(2\pi k+t)x} + O\biggl(\frac{\ln|k|}{k}\biggr), \tag{37}$$

where v^*_j is the eigenvector of the matrix C^* corresponding to $\overline{\mu_j}$. Since $(v_j, v^*_j) \neq 0$, it follows that (37) and (34) contradict (36). Thus, $\lambda_{k,j}$ is a simple eigenvalue, $r_j = 1$, and relation (33) follows from assertion (a).

Let us prove assertion (c). By (a), all sufficiently large (in absolute value) eigenvalues of the operator $L_t(Q)$ lie inside disks of radius $O(k^{-1}\ln|k|)$ centered at the points $\mu_{k,j} = (2\pi k + t)^2 + \mu_j$. Let us prove that, asymptotically, there is only one eigenvalue inside these disks. Suppose that there are two different eigenvalues satisfying relation (33) for a fixed j and a fixed sufficiently large (in absolute value) k. It has already been proved that to these two eigenvalues there correspond eigenfunctions of the form

$$\Psi_p(x) = v_j e^{i(2\pi k+t)x} + O\biggl(\frac{\ln|k|}{k}\biggr), \qquad p = 1, 2.$$

To the complex conjugate eigenvalues of the adjoint operator there correspond eigenfunctions of the form

$$\Psi^*_p(x) = v^*_j e^{i(2\pi k+t)x} + O\biggl(\frac{\ln|k|}{k}\biggr), \qquad p = 1, 2.$$

Since the eigenvalues are distinct, we have

$$0 = (\Psi_1(x), \Psi^*_2(x)) = (v_j, v^*_j) + O\biggl(\frac{\ln|k|}{k}\biggr) = 1 + O\biggl(\frac{\ln|k|}{k}\biggr),$$

which cannot be true for large $|k|$.

It remains to prove that the root functions $\Psi_{k,j,p}(x)$ of the operator $L_t(Q)$ form a Riesz basis in the space $L^m_2(0,1)$. For sufficiently large $|k| > N$, there are no associated functions; therefore, we can drop the index p. It follows from the asymptotics of (34) that, for any function $f(x) \in L^m_2(0,1)$, the following series is convergent:

$$\sum_{j=1}^{m} \sum_{|k|>N+1}^{\infty} |(f, \Psi_{k,j}(x))|^2 < \infty.$$

By Bari's definition [11, Chap. 6], this implies that the system of eigenfunctions of the operator under consideration is Bessel. The number of associated functions is finite; therefore, the system of all root functions of the operator is also Bessel. The system of root functions of the adjoint operator has the asymptotics (37); therefore, it is also Bessel. Obviously, the system of $\chi_{k,j,p}(x)$ biorthogonal to the system of root functions $\Psi_{k,j,p}(x)$ for $|k| > N$ is of the form (the index p is dropped)

$$\chi_{k,j}(x) = \frac{\Psi^*_{k,j}(x)}{(\Psi_{k,j}(x), \Psi^*_{k,j}(x))}\,.$$

It follows from (34) and (37) that, for $|k| > N$,

$$(\Psi_{k,j}(x), \Psi^*_{k,j}(x)) = (v_j, v^*_j) + O(k^{-1} \ln |k|) > C_7;$$

therefore, the biorthogonal system is also Bessel. It is well known [2, Chap. 2] that the systems of root functions of the operators $L_t(Q)$ and $L^*_t(Q)$ are complete in the space $L^m_2(0,1)$. By Bari's theorem, if two biorthogonal, almost normalized systems are complete and Bessel, then they both are Riesz bases. This concludes the proof of the theorem. □

REFERENCES

1. M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations* (Scottish Academic Press, Edinburgh–London, 1973).
2. M. A. Naimark, *Linear Differential Operators* (Nauka, Moscow, 1969) [in Russian].
3. R. Carlson, "Large eigenvalues and trace formulas for matrix Sturm–Liouville problems," SIAM J. Math. Anal. **30** (5), 949–962 (1999).
4. O. A. Veliev and F. G. Maksudov, "Non-self-adjoint differential operators in the space of vector functions with periodic coefficients," Dokl. Akad. Nauk SSSR **258**, 26–30 (1981) [Soviet Math. Dokl., **23**, 476–478 (1981)].
5. O. A. Veliev and F. G. Maksudov, "Spectral analysis of differential operators with periodic matrix coefficients," Differentsial'nye Uravneniya **25** (3), 400–409 (1989) [Differential Equations, **25** (3), 271–277 (1989)].
6. A. A. Shkalikov, "On the Riesz basis property of the root vectors of ordinary differential operators," Uspekhi Mat. Nauk **34** (5), 235–236 (1979) [Russian Math. Surveys **34** (5), 249–250 (1979)].
7. A. A. Shkalikov, "On the basis property of the eigenfunctions and associated functions of ordinary differential operators with integral boundary conditions," Vestnik Moskov. Univ. Ser. I Mat. Mekh., No. 6, 12–21 (1982).
8. A. A. Shkalikov, "Boundary-value problems for ordinary differential equations with spectral parameter in the boundary conditions," Trudy Sem. Petrovsk., No. 9, 190–229 (1983).
9. A. A. Shkalikov, "On the Riesz basis property of the eigenfunctions and associated functions of ordinary differential operators in the space of vector functions," in *Functional Analysis and Its Applications in Mechanics and Probability Theory* (Izd. Moskov. Univ., Moscow, 1984) [in Russian].
10. L. M. Luzhina, "Regular spectral problems in the spaces of vector functions," Vestnik Moskov. Univ. Ser. I Mat. Mekh. No. 1, 31–35 (1988).
11. I. Ts. Gokhberg and M. G. Krein, *Introduction in the Theory of Linear Non-Self-Adjoint Operators in Hilbert Space* (Nauka, Moscow, 1965) [in Russian].