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On Quasi-Recurrent Spaces With Ricci Quarter-Symmetric Metric Connection

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Abstract

In [3], Mishra and Pandey defined Ricci quarter-symmetric metric connection in Riemanian manifold. In [5],Uysal and Doğan defined D-recurrent spaces with semi-symmetric metric connection and constructed an example of these spaces. In these paper we define quasi-recurrent spaces with Ricci quarter- symmetric metric connection and establish an example of such spaces.

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1 Introduction

A non-flat Riemannian space M^n $(n > 2)$ is called a quasi-recurrent space if the curvature tensor R_{kjih} satisfies the condition

$$\nabla_l R_{kjih} = a_l R_{kjih} + b_k R_{ljih} + c_j R_{klih} + d_i R_{kjlh} + e_h R_{kjil} \tag{1.1}$$

where a, b, c, d, e are 1-forms (non-zero simultaneously) and ∇ denotes the covariant differentiation with respect to Levi Civita connection $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$. a, b, c, d, e are called the associated 1-forms of the space and n-dimensional space of this kind is denoted by $(QR)_n$.

Let (M^n, g) be an n-dimensional differentiable manifold of class C^∞ with metric tensor g and Levi Civita connection ∇. A linear connection D on (M^n, g) is said
quarter-symmetric metric to be a Ricci Quarter-Symmetric Metric Connection if the torsion tensor T of the connection D and the metric tensor g of the manifold satisfy [2], [6],

$$T^h_{jk} = R^h_j w_k - R^h_k w_j\,, \qquad D_k g_{ji} = 0\,, \tag{1.2}$$

where w is a 1-form associated with the torsion tensor of the connection D and $R^h_j = R_{ja} g^{ah}$, R_{ij} is the Ricci tensor of the Levi Civita connection ∇.

Then we have

$$\Gamma^h_{ji} = \left\{ \begin{array}{c} h \\ ji \end{array} \right\} + R^h_j w_i - R_{ji} w^h, \tag{1.3}$$

where Γ^h_{ji} are the connection coefficients of the Ricci quarter-symmetric metric connection and $w^h = w_t\, g^{th}$.

The curvature tensor L^h_{kji} of the manifold M^n is defined in [6] by,

$$L^h_{kji} = \partial_k \Gamma^h_{ji} - \partial_j \Gamma^h_{ki} + \Gamma^h_{ka}\Gamma^a_{ji} - \Gamma^h_{ja}\Gamma^a_{ki}\,, \qquad \left(\partial_k = \frac{\partial}{\partial x^k}\right) \tag{1.4}$$

and the curvature tensor of the Riemannian manifold

$$R^h_{kji} = \partial_k \left\{ \begin{array}{c} h \\ ji \end{array} \right\} - \partial_j \left\{ \begin{array}{c} h \\ ki \end{array} \right\} + \left\{ \begin{array}{c} h \\ ka \end{array} \right\}\left\{ \begin{array}{c} a \\ ji \end{array} \right\} - \left\{ \begin{array}{c} h \\ ja \end{array} \right\}\left\{ \begin{array}{c} a \\ ki \end{array} \right\}, \tag{1.5}$$

then substituting (1.3) in (1.4), we obtain the following equation for the curvature tensor L^h_{kji} of (M^n, g, D).

$$\begin{aligned} L_{kjih} &= R_{kjih} - R_{kh}\, w_{ji} + R_{jh}\, w_{ki} - w_{kh}\, R_{ji} + w_{jh}\, R_{ki} \\ &\quad + (\nabla_k R_{jh} - \nabla_j R_{kh})\, w_i - (\nabla_k R_{ji} - \nabla_j R_{ki})\, w_h\,, \end{aligned} \tag{1.6}$$

where we put

$$w_{ji} \,=\, \nabla_j\, w_i - R_{jt}\, w^t\, w_i \,+\, \frac{1}{2}\, w_t\, w^t\, R_{ji}\,; \quad w^h_k \,=\, w_{ka}\, g^{ah}. \tag{1.7}$$

The curvature tensor of the Ricci quarter-symmetric metric connection L_{kjih} satisfies the following properties:

$$\begin{aligned} &(i) \quad L_{kjih} = -L_{jkih}, \\ &(ii) \quad L_{kjih} = -L_{kjhi}, \\ &(iii) \quad L_{kjih} = L_{jkhi}, \\ &(iv) \quad L_{kkih} \,=\, L_{kjhh} \,=\, 0. \end{aligned} \tag{1.8}$$

Definition 1.1. An n-dimensional, ($n > 2$), quasi-recurrent space with Ricci quarter-symmetric metric connection is a non-flat space satisfying the condition

$$D_l L_{kjih} = a_l L_{kjih} + b_k L_{ljih} + c_j L_{klih} + d_i L_{kjlh} + e_h L_{kjil}, \tag{1.9}$$

where L_{kjih} is the curvature tensor of the space and a, b, c, d, e are 1-forms (non-zero simultaneously). Such spaces will be represented by $((QR)_n, D)$ in short.

Theorem ,. Five associated 1-forms cannot be all different.

Proof; By using the following method which is used in [1], we obtain that $b = c$ and $d = e$.

Interchanging the indices k and j in (1.9) we obtain

$$D_l L_{jkih} = a_l L_{jkih} + b_j L_{lkih} + c_k L_{jlih} + d_i L_{jklh} + e_h L_{jkil}. \tag{1.10}$$

Adding (1.9) and (1.10) and using (1.8i), we get

$$(b_k - c_k) L_{ljih} + (b_j - c_j) L_{lkih} = 0 \tag{1.11}$$

or

$$A_k L_{ljih} + A_j L_{lkih} = 0 \tag{1.12}$$

where $A_k = (b_k - c_k)$.

We want to show that $A_k = 0$ ($k = 1, 2, ..., n$).

On the contrary assume that there exists a fixed q for which $A_q \neq 0$. If we set $k = j = q$ in (1.12), then we find that $A_q L_{lqih} + A_q L_{lqih} = 0$ which means that the curvature tensor $L_{lqih} = 0$ for all l, i, h.

On the other hand, if we take $k = q$ in (1.12), we obtain that $A_q L_{ljih} + A_j L_{lqih} = 0$.

Since $A_q \neq 0$, then the curvature tensor $L_{ljih} = 0$ for all l, j, i, h, which contradicts with the hypothesis. Therefore, $A_k = 0$, which means that $b_k = c_k$

for all k.

Interchanging the indices i and h in (1.9) we obtain

$$D_l L_{kjhi} = a_l L_{kjhi} + b_k L_{ljhi} + c_j L_{klhi} + d_h L_{kjli} + e_i L_{kjhl}. \quad (1.13)$$

If we use the same method for indices i and h, we find that $d_i = e_i$.

Furthermore the condition (1.9) can be expressed in

$$D_l L_{kjih} = a_l L_{kjih} + b_k L_{ljih} + b_j L_{klih} + d_i L_{kjlh} + d_h L_{kjil}. \quad (1.14)$$

So a non-flat space with Ricci quarter-symmetric metric connection is called quasi-recurrent if its curvature tensor L_{kjih} satisfies the condition (1.14).

If, in particular, $b_i = 0$ and $d_i = 0$ in (1.9), then we have,

Definition 1.2. The space (M, g, D) is called $D - recurrent$, if there exists a covariant vector field $a_l \neq 0$ such that

$$D_l L^h_{kji} = a_l\, L^h_{kji}. \quad (1.15)$$

Moreover, if $R^h_j = \delta^h_j$ in (1.2),then we obtain the results in [5].

2 An Example of Quasi-Recurrent Spaces With Ricci Quarter-Symmetric Metric Connection

We define the metric g in the coordinate space R^n $(n \geq 4)$ by the formula,[1] [4]

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \quad (2.1)$$

where $[k_{\alpha\beta}]$ is a symmetric non - singular matrix consisting of constants and φ is independent of x^n. The only non-zero components of the Christoffel' s symbols $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$, the Riemannian curvature tensor R_{kjih} and the Ricci tensor R_{ji} are [4]

$$\left\{ \begin{array}{c} \lambda \\ 11 \end{array} \right\} = -\frac{1}{2} k^{\lambda\beta} \varphi_{.\beta}\ , \quad \left\{ \begin{array}{c} n \\ 11 \end{array} \right\} = \frac{1}{2} \varphi_{.1}\ , \quad \left\{ \begin{array}{c} n \\ 1\alpha \end{array} \right\} = \frac{1}{2} \varphi_{.\alpha} \quad (2.2)$$

[1] In this section, let each Latin indices run over $1, 2, ..., n$ and each Greek indices run over $2, 3, ..., n-1$.

and

$$R_{1\alpha\beta 1} = \frac{1}{2}\varphi_{.\alpha\beta}\,, \quad R_{11} = \frac{1}{2}k^{\alpha\beta}\varphi_{.\alpha\beta}, \tag{2.3}$$

respectively, where (.) denotes the partial differentiation with respect to coordinates and $k^{\alpha\beta}$ is the inverse matrix of $k_{\alpha\beta}$.

For the metric (2.1), we consider $k_{\alpha\beta} = \delta_{\alpha\beta}$ and

$$\varphi = \frac{1}{2}\sin(k_{\alpha\beta}\,\frac{\pi}{2})x^{\alpha}\,x^{\beta}\,e^{x^1}. \tag{2.4}$$

Considering the metric, the only non-zero components of the curvature tensor R_{kjih} and the Ricci tensor R_{ji} are

$$R_{1\alpha\alpha 1} = \frac{e^{x^1}}{2} \qquad\qquad R_{11} = (n-2)\frac{e^{x^1}}{2}. \tag{2.5}$$

Now, differentiating covariantly R_{kjih} and R_{ji} with respect to all components $(l = 1, 2, ..., n)$, we get the non-zero components of $\nabla_l R_{kjih}$ and $\nabla_l R_{ji}$ as

$$\nabla_1 R_{1\alpha\alpha 1} = \frac{e^{x^1}}{2} \qquad\qquad \nabla_1 R_{11} = (n-2)\frac{e^{x^1}}{2}. \tag{2.6}$$

For a chosen α, let

$$w_h = \begin{cases} \psi(x^{\alpha}), & h = \alpha \\ 0, & otherwise \end{cases} \tag{2.7}$$

where $\psi(x^{\alpha})$ is a continuous functions of x^{α} on the interval $I = [a, b]$.

We obtain the non-zero components of Γ^h_{ji} and w_{ji} according to the our assumption (2.7) as follows

$$\begin{aligned}
\Gamma^{\alpha}_{11} &= \left\{ \begin{matrix} \alpha \\ 11 \end{matrix} \right\} + R^{\alpha}_{1}w_1 - R_{11}w^{\alpha} = \left\{ \begin{matrix} \alpha \\ 11 \end{matrix} \right\} - R_{11}w_{\alpha} = \left\{ \begin{matrix} \alpha \\ 11 \end{matrix} \right\} - (n-2)\,\frac{e^{x^1}}{2}\,\psi(x^{\alpha}) \\
\Gamma^{n}_{11} &= \left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} + R^{n}_{1}w_1 - R_{11}w^{n} = \left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} \\
\Gamma^{n}_{1\alpha} &= \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} + R^{n}_{1}w_{\alpha} - R_{1\alpha}w^{n} = \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} + R_{11}w_{\alpha} = \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} + (n-2)\,\frac{e^{x^1}}{2}\,\psi(x^{\alpha}) \\
\Gamma^{n}_{\alpha 1} &= \left\{ \begin{matrix} n \\ \alpha 1 \end{matrix} \right\} + R^{n}_{\alpha}w_1 - R_{\alpha 1}w^{n} = \left\{ \begin{matrix} n \\ \alpha 1 \end{matrix} \right\}
\end{aligned} \tag{2.8}$$

$$w_{11} = \nabla_1 w_1 - R_{1t}w^t w_1 + \frac{1}{2}w_t w^t R_{11} = \frac{\partial w_1}{\partial x^1} - w_t\left\{ \begin{array}{c} t \\ 11 \end{array} \right\} - R_{1t}w^t w_1 + \frac{1}{2}w_\alpha w^\alpha R_{11}$$
$$= -\psi(x^\alpha)\left\{ \begin{array}{c} \alpha \\ 11 \end{array} \right\} + \frac{1}{4(n-2)}e^{x^1} g^{\alpha\beta} w_\alpha w_\beta \tag{2.9}$$

$$w_{\alpha\alpha} = \nabla_\alpha w_\alpha - R_{\alpha t}w^t w_\alpha + \frac{1}{2}w_t w^t R_{\alpha\alpha} = \frac{\partial w_\alpha}{\partial x^\alpha} - w_t\left\{ \begin{array}{c} t \\ \alpha\alpha \end{array} \right\} - R_{\alpha t}w^t w_\alpha + \frac{1}{2}w_\alpha w^\alpha R_{\alpha\alpha}$$
$$= \psi'(x^\alpha) \tag{2.10}$$

Using (1.6), (2.5), (2.6), (2.7), (2.9) and (2.10), we get the only non-zero component of the curvature tensor L_{kjih},

$$\begin{aligned} L_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - R_{11}w_{\alpha\alpha} + R_{\alpha 1}w_{1\alpha} - w_{11}R_{\alpha\alpha} + w_{\alpha 1}R_{1\alpha} \\ &+ (\nabla_1 R_{\alpha 1} - \nabla_\alpha R_{11})w_\alpha - (\nabla_1 R_{\alpha\alpha} - \nabla_\alpha R_{1\alpha})w_1 \\ &= R_{1\alpha\alpha 1} - R_{11}w_{\alpha\alpha} \\ &= \frac{1}{2}\left[1 - (n-2)\,\psi'(x^\alpha)\right] e^{x^1}. \end{aligned} \tag{2.11}$$

Using the properties of the curve L_{kjih}, we have other non-zero components of the curvature tensor L_{kjih}

$$\begin{aligned} L_{\alpha 1\alpha 1} &= -L_{1\alpha\alpha 1} = -\frac{e^{x^1}}{2}\left[1 - (n-2)\psi'(x^\alpha)\right] \\ L_{1\alpha 1\alpha} &= -L_{1\alpha\alpha 1} = -\frac{e^{x^1}}{2}\left[1 - (n-2)\psi'(x^\alpha)\right] \\ L_{\alpha 11\alpha} &= L_{1\alpha\alpha 1} = \frac{e^{x^1}}{2}\left[1 - (n-2)\psi'(x^\alpha)\right]. \end{aligned} \tag{2.12}$$

Hence, we get the non-zero components of $D_l L_{kjih}$

$$\begin{aligned} D_1 L_{1\alpha\alpha 1} &= \frac{\partial L_{1\alpha\alpha 1}}{\partial x^1} - (L_{s\alpha\alpha 1} + L_{1\alpha\alpha s})\,\Gamma^s_{11} - (L_{1s\alpha 1} + L_{1\alpha s1})\,\Gamma^s_{1\alpha} \\ &= \frac{e^{x^1}}{2}\left[1 - (n-2)\psi'(x^\alpha)\right] \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} D_\alpha L_{1\alpha\alpha 1} &= \frac{\partial L_{1\alpha\alpha 1}}{\partial x^\alpha} - (L_{s\alpha\alpha 1} + L_{1\alpha\alpha s})\,\Gamma^s_{\alpha 1} - (L_{1s\alpha 1} + L_{1\alpha s1})\,\Gamma^s_{\alpha\alpha} \\ &= -\frac{e^{x^1}}{2}(n-2)\psi''(x^\alpha). \end{aligned} \tag{2.14}$$

We want to show that (M^n, g, D) is $((QR)_n, D)$. Let us consider the 1-forms

$$a_i = \begin{cases} \frac{1}{3}, & i = 1 \\ 1, & i = \alpha \\ 0, & otherwise \end{cases} \quad b_i = \begin{cases} \frac{1}{3}, & i = 1 \\ \tanh(x^\alpha), & i = \alpha \\ 0, & otherwise \end{cases} \quad \text{and} \quad d_i = \begin{cases} \frac{1}{3}, & i = 1 \\ 0, & otherwise \end{cases} \tag{2.15}$$

In order to verify the condition (1.14), it is sufficient to check the following relations:

$$\begin{aligned}
&(A)\ D_1 L_{1\alpha\alpha 1} = a_1 L_{1\alpha\alpha 1} + b_1 L_{1\alpha\alpha 1} + b_\alpha L_{11\alpha 1} + d_\alpha L_{1\alpha 11} + d_1 L_{1\alpha\alpha 1} \\
&(B)\ D_\alpha L_{11\alpha 1} = a_\alpha L_{11\alpha 1} + b_1 L_{\alpha 1\alpha 1} + b_1 L_{1\alpha\alpha 1} + d_\alpha L_{11\alpha 1} + d_1 L_{11\alpha\alpha} \\
&(C)\ D_\alpha L_{1\alpha 11} = a_\alpha L_{1\alpha 11} + b_1 L_{\alpha\alpha 11} + b_\alpha L_{1\alpha 11} + d_1 L_{1\alpha\alpha 1} + d_1 L_{1\alpha 1\alpha} \\
&(D)\ D_\alpha L_{1\alpha\alpha 1} = a_\alpha L_{1\alpha\alpha 1} + b_1 L_{\alpha\alpha\alpha 1} + b_\alpha L_{1\alpha\alpha 1} + d_\alpha L_{1\alpha\alpha 1} + d_1 L_{1\alpha\alpha\alpha} \\
&(E)\ D_1 L_{\alpha\alpha\alpha 1} = a_1 L_{\alpha\alpha\alpha 1} + b_\alpha L_{1\alpha\alpha 1} + b_\alpha L_{\alpha 1\alpha 1} + d_\alpha L_{\alpha\alpha 11} + d_1 L_{\alpha\alpha\alpha 1} \\
&(F)\ D_1 L_{1\alpha\alpha\alpha} = a_1 L_{1\alpha\alpha\alpha} + b_1 L_{1\alpha\alpha\alpha} + b_\alpha L_{11\alpha\alpha} + d_\alpha L_{1\alpha 1\alpha} + d_\alpha L_{1\alpha\alpha 1}
\end{aligned}$$

For the other cases, (1.14) holds trivially.

From (1.8 iv), (2.13) and (2.15), we get the following relation for the right-hand side (r.h.s) and the left-hand side (l.h.s) of (A):

$$\text{r.h.s of (A)} = (a_1 + b_1 + d_1) L_{1\alpha\alpha 1} = \frac{1}{2} [1 - (n-2)\,\psi'(x^\alpha)]\, e^{x^1} = D_1 L_{1\alpha\alpha 1} = \text{l.h.s of (A)}$$

where $(')$ denotes the derivative with respect to x^α.

Using the relations of (1.8 i) and (1.8 iv), we obtain that

$$\text{r.h.s. of (B)} = b_1 (L_{\alpha 1\alpha 1} + L_{1\alpha\alpha 1}) = 0.$$

By using the similar argument in (B), it is easy to see that the relation (C) is also true.

From (1.8 iv), (2.14) and (2.15), we get the following relation for the right-hand side (r.h.s) and the left-hand side (l.h.s) of (D):

$$\begin{aligned}\text{r.h.s of (D)} &= (a_\alpha + b_\alpha + d_\alpha) L_{1\alpha\alpha 1} \\ &= \frac{1}{2}\,[1\,+\,\tanh(x^\alpha)]\,[1\,-\,(n-2)\,\psi'(x^\alpha)\,]\,e^{x^1} \\ &= -\frac{e^{x^1}}{2}\,(n-2)\psi''(x^\alpha) \\ &= D_\alpha L_{1\alpha\alpha 1} = \text{l.h.s of (D)}\end{aligned}$$

From the above equation, we have the second order differential equation

$$\psi''(x^\alpha) = [1\,+\,\tanh(x^\alpha)]\,[\psi'(x^\alpha)\,-\,\frac{1}{n-2}] \tag{2.16}$$

which has the solution

$$\psi(x^\alpha)\;=\;(\frac{1}{n-2}\;+\;\frac{c_1}{2}\,)\,x^\alpha\;+\;\frac{c_1}{4}\,e^{2\,x^\alpha}\;+\;c_2 \tag{2.17}$$

with arbitrary constant c_1 and c_2.

Again using the relations (1.8 i) and (1.8 iv), we obtain that

$$\text{r.h.s. of (E)} = b_\alpha\,(L_{1\alpha\alpha 1}\,+\,L_{\alpha 1\alpha 1}) = 0.$$

By using the similar argument in (E), it can be shown that the relation (F) is also true.

Finally we have determined the functions $\psi(x^\alpha)$, therefore the coefficients of the connection are determined. Hence, R^n with the metric (2.1) is a $((QR)_n, D)$.

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