On the Basis Property of the Root Functions of Sturm-Liouville Operators with General Regular Boundary Conditions.

Cemile Nur and O. A. Veliev
Department of Mathematics, Dogus University, Kadiköy, Istanbul, Turkey.
E-mail: cnur@dogus.edu.tr; oveliev@dogus.edu.tr

Abstract

We obtain the asymptotic formulas for the eigenvalues and eigenfunctions of the Sturm-Liouville operators with general regular boundary conditions. Using these formulas, we find sufficient conditions on the potential q such that the root functions of these operators do not form a Riesz basis.

Key Words: Asymptotic formulas, Regular boundary conditions. Riesz basis.
AMS Mathematics Subject Classification: 34L05, 34L20.

1 Introduction and Preliminary Facts

In this paper we consider the operators generated in $L_2[0,1]$ by the differential expression

$$l\left(y\right)=-y''+q(x)y \tag{1}$$

and regular boundary conditions that are not strongly regular. Note that, if the boundary conditions are strongly regular, then the root functions (eigenfunctions and associated functions) form a Riesz basis (this result was proved independently in [6], [9] and [17]). In the case when an operator is regular but not strongly regular, the root functions generally do not form even usual basis. However, Shkalikov [20, 21] proved that they can be combined in pairs, so that the corresponding 2-dimensional subspaces form a Riesz basis of subspaces.

To describe the results of this paper and preliminary results let us classify all regular boundary conditions that are not strongly regular. One can readily see from pages 62-63 of [18] that all regular boundary conditions that are not strongly regular can be written in the form

$$\begin{aligned} a_1y_0'+b_1y_1'+a_0y_0+b_0y_1&=0,\\ c_0y_0+d_0y_1&=0, \end{aligned} \tag{2}$$

if

$$b_1c_0+a_1d_0\neq 0. \tag{3}$$

and $\theta_0^2-4\theta_1\theta_{-1}=0$, where , a_i,b_i,c_0,d_0, $i=0,1$, are complex numbers and θ_0,θ_1 and θ_{-1} are defined by

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s=w_1\left(b_1c_0+a_1d_0\right)\left(s+\frac{1}{s}\right)+2\left(a_1c_0+b_1d_0\right)w_1 \tag{4}$$

in p.63 of [18]. Thus, by (4), $\theta_{-1}=\theta_1=w_1\left(b_1c_0+a_1d_0\right),\ \theta_0=2\left(a_1c_0+b_1d_0\right)w_1$, and hence the equality $\theta_0^2-4\theta_1\theta_{-1}=0$ implies that

$$4\omega_1^2\left[\left(a_1c_0+b_1d_0\right)^2-\left(b_1c_0+a_1d_0\right)^2\right]=0,$$

that is, $\left(a_1^2-b_1^2\right)\left(c_0^2-d_0^2\right)=0$ which means that at least one of the following conditions holds:

$$a_1=\pm b_1,\ \ c_0=\pm d_0.$$

First suppose that $a_1=(-1)^\sigma b_1$, where $\sigma=0,1$. This with (3) implies that both a_1 and b_1 are not zero and at least one of c_0 and d_0 is not zero. If $c_0\neq 0$, then (2) can be written in the form

$$\begin{aligned} y_0'+(-1)^\sigma y_1'+\alpha_1 y_1&=0,\\ y_0+\alpha_2 y_1&=0, \end{aligned}\tag{5}$$

where $\alpha_1=\dfrac{b_0}{a_1}-\dfrac{a_0d_0}{a_1c_0}$, $\alpha_2=\dfrac{d_0}{c_0}$, $a_1,c_0\neq 0$ and $\alpha_2\neq-(-1)^\sigma$ due to (3).

Similarly, if $d_0\neq 0$, then (2) can be transformed to

$$\begin{aligned} y_0'+(-1)^\sigma y_1'+\alpha_3 y_0&=0,\\ \alpha_4 y_0+y_1&=0, \end{aligned}\tag{6}$$

where $\alpha_3=\dfrac{a_0}{a_1}-\dfrac{b_0c_0}{a_1d_0}$, $\alpha_4=\dfrac{c_0}{d_0}$, $a_1,d_0\neq 0$ and by (3) $\alpha_4\neq-(-1)^\sigma$.

Now suppose that $d_0=(-1)^\sigma c_0$. Arguing as in the reductions of (5) and (6) we arrive at the boundary conditions

$$\begin{aligned} y_0'+\beta_1 y_1'+\beta_2 y_1&=0,\\ y_0+(-1)^\sigma y_1&=0, \end{aligned}\tag{7}$$

where $\beta_1=\dfrac{b_1}{a_1}$, $\beta_2=\left(\dfrac{b_0}{a_1}\mp\dfrac{a_0}{a_1}\right)$, $a_1,c_0\neq 0$ and

$$\beta_1\neq-(-1)^\sigma\tag{8}$$

and the boundary conditions

$$\begin{aligned} \beta_3 y_0'+y_1'+\beta_4 y_1&=0,\\ y_0+(-1)^\sigma y_1&=0, \end{aligned}\tag{9}$$

where $\beta_3=\dfrac{a_1}{b_1}$, $\beta_4=\dfrac{b_0}{b_1}\mp\dfrac{a_0}{b_1}$, $b_1,c_0\neq 0$ and

$$\beta_3\neq-(-1)^\sigma\tag{10}$$

for $\sigma=0,1$.

One can verify in the standard way that, the boundary conditions (5) and (6), are the adjoint boundary conditions to (9) and (7), respectively, where $\alpha_3=-(-1)^\sigma\overline{\beta_2}$, $\alpha_4=\overline{\beta_1}$ and $\alpha_1=(-1)^\sigma\overline{\beta_4}$, $\alpha_2=\overline{\beta_3}$.

Thus to consider all regular boundary conditions that are not strongly regular it is enough to investigate the boundary conditions (7) and (9). Note that these boundary conditions depend on two parameters. Let us describe the special cases that were investigated.

Case (a) The cases $\beta_2, \beta_4 = 0$, $\beta_1, \beta_3 = (-1)^\sigma$ in (7), (9) for $\sigma = 1$ and $\sigma = 0$ coincide with the periodic and antiperiodic boundary conditions respectively. These boundary conditions are the ones more commonly studied. Therefore, let us briefly describe some historical developments related to the Riesz basis property of the root functions of the periodic and antiperiodic boundary value problems. First results were obtained by Kerimov and Mamedov [8]. They established that, if

$$q \in C^4[0,1], \; q(1) \neq q(0),$$

then the root functions of the operator $L(q)$ form a Riesz basis in $L_2[0,1]$, where $L(q)$ denotes the operator generated by (1) and the periodic boundary conditions.

The first result in terms of the Fourier coefficients of the potential q was obtained by Dernek and Veliev [1]. Makin [11] extended this result for the larger class of functions. Shkalilov and Veliev obtained in [22] more general results which cover all results about periodic and antiperiodic boundary conditions discussed above.

The other interesting results about periodic and antiperiodic boundary conditions were obtained in [2-5, 7, 14-16, 23-25].

Case (b) The cases $\beta_2, \beta_4 \neq 0$ and $\beta_1, \beta_3 = (-1)^\sigma$ are investigated in [12, 13] and it was proved that the system of the root functions of the Sturm-Liouville operator corresponding to this case is a Riesz basis in $L_2(0,1)$ (see Theorem 1 of [12,13]).

Case (c) The cases $\beta_2, \beta_4 = 0$ and $\beta_1, \beta_3 \neq (-1)^\sigma$ are investigated in [12, 13] and [19].

We call the boundary conditions (7) and (9) for $\beta_2, \beta_4 \neq 0$ and $\beta_1, \beta_3 \neq (-1)^\sigma$ which are different from the special cases (a) , (b) and (c) as general regular boundary conditions that are not strongly regular. Note that in any case $\beta_1, \beta_3 \neq -(-1)^\sigma$ by (8) and (10). For the case (c) and general boundary conditions Makin [12, 13] proved that the systems of the root functions of the Sturm-Liouville operators corresponding to these cases are Riesz bases in $L_2(0,1)$ if and only if all large eigenvalues are multiple. Note that this result is not effective, since the conditions are given in implicit form and can not be verified for concrete potentials. In [19] we find explicit conditions on potential such that the system of the root functions of the Sturm-Liouville operator corresponding to the case (c) does not form a Reisz basis. Namely we proved that if

$$\lim_{n\to\infty} \frac{\ln|n|}{n s_{2n}} = 0, \tag{11}$$

where $s_n = (q, \sin 2\pi n t)$ and $(.,.)$ is the inner product in $L_2[0,1]$, then the large eigenvalues of each of the operators corresponding to these cases are simple for $\sigma = 1$. Moreover, if there exists a sequence $\{n_k\}$ such that (11) holds when n is replaced by n_k, then the root functions of these operators do not form a Riesz basis. Similarly, if the condition

$$\lim_{n\to\infty} \frac{\ln|n|}{n s_{2n+1}} = 0$$

holds instead of (11), then the same statements continue to hold for $\sigma = 0$.

In this paper we find explicit conditions on potential q such that the system of the root functions of the Sturm-Liouville operator generated by (1) and the general regular boundary conditions does not form a Reisz basis.

Now let us describe briefly the main results of this paper. Let $T_1^\sigma(q)$ and $T_2^\sigma(q)$ be the Sturm-Liouville operators associated by the boundary conditions (7) and (9) respectively. Without loss of generality we assume that

$$\int_0^1 q(t)\,dt = 0.$$

First we prove that if $q \in L_1[0,1]$ and

$$\int_0^1 \sin(2\pi nt)\, q(t)\, dt = o\left(\frac{1}{n}\right) \tag{12}$$

then the large eigenvalues of $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 1$, are simple. Moreover if there exists a subsequence $\{n_k\}$ such that (12) holds whenever n is replaced by n_k, then the system of the root functions of each operators $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 1$, does not form a Riesz basis. The same results continue to hold for $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 0$, if instead of (12) the condition

$$\int_0^1 \sin((2n+1)\pi t)\, q(t)\, dt = o\left(\frac{1}{n}\right) \tag{12a}$$

holds.

The other main result is the following: If the potential q is an absolutely continuous function and

$$q(0) + (-1)^\sigma q(1) \neq \frac{2\beta_2^2}{1-\beta_1^2} \tag{13}$$

then the large eigenvalues of $T_1^\sigma(q)$ for $\sigma = 0, 1$ are simple and the system of the root functions of $T_1^\sigma(q)$ does not form a Riesz basis. Similarly, if the condition

$$q(0) + (-1)^\sigma q(1) \neq \frac{2\beta_4^2}{\beta_3^2 - 1} \tag{14}$$

holds instead of (13), then the same results remain valid for $T_2^\sigma(q)$ for $\sigma = 0, 1$. Moreover we obtain subtle asymptotic formulas for the eigenvalues and eigenfunctions for the operators $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for both cases $q \in L_1[0,1]$ and q is an absolutely continuous function.

Note that the general cases we investigate in this paper are essentially different from the case (c) as the method of investigations and obtained results.

2 Main Results

We will focus only on the operator $T_1^1(q)$. The investigations of the operators $T_1^0(q), T_2^0(q)$ and $T_2^1(q)$ are similar. First let us prove the following simple proposition about $T_1^1(0)$. Note that the simplest case $q(x) \equiv 0$ was completely solved in [10]. Here we write the asymptotic formulas for the eigenvalues of $T_1^1(0)$ in the form we need.

Proposition 1 *The square roots (with nonnegative real part) of the eigenvalues of the operator $T_1^1(0)$ consist of the sequences $\{\mu_{n,1}(0)\}$ and $\{\mu_{n,2}(0)\}$ satisfying*

$$\mu_{n,1}(0) = 2\pi n, \tag{15}$$

$$\mu_{n.2}(0) = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + O\left(\frac{1}{n^2}\right). \tag{16}$$

Proof. Using the fundamental solutions $e^{i\mu x}$ and $e^{-i\mu x}$ of $-y'' = \lambda y$ where $\mu = \sqrt{\lambda}$, one can readily see that the characteristic determinant $\Delta_0(\mu)$ of $T_1^1(0)$ has the form

$$\Delta_0(\mu) = \left(1 - e^{i\mu}\right)\left(i\mu + \beta_1 i\mu e^{-i\mu} - \beta_2 e^{-i\mu}\right) + \left(i\mu + \beta_1 i\mu e^{i\mu} + \beta_2 e^{i\mu}\right)\left(1 - e^{-i\mu}\right) = 0.$$

After simplifying this equation, we have

$$\Delta_0(\mu) = \left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \beta_2\left(e^{i\mu} + 1\right)\right] = 0 \tag{17}$$

which is equivalent to

$$1-e^{-i\mu}=0 \text{ or } f(\mu)=0 \tag{18}$$

where

$$f(\mu)=e^{i\mu}-1-\frac{i\beta_2}{\beta_1-1}\frac{e^{i\mu}+1}{\mu}=e^{i\mu}-1+O\left(\frac{1}{\mu}\right) \tag{19}$$

The solution of the first equation in (18) is $\mu_{n,1}(0)=2\pi n$ for $n\in\mathbb{Z}$, that is, (15) is proved.

To prove (16), we estimate the roots of (19). Using Rouche's theorem on the circle $\left\{\mu: \ |\mu-2\pi n|=\dfrac{c}{n}\right\}$ for some constant c, one can easily see that, the roots of (19) has the form

$$\mu_{2,n}^{0}=2\pi n+\xi \ \& \ \xi=O\left(\frac{1}{n}\right). \tag{20}$$

Now we prove that

$$\xi=\frac{\beta_2}{\beta_1-1}\frac{1}{\pi n}+O\left(\frac{1}{n^2}\right). \tag{21}$$

For this, let us consider the roots of (19) in detail. By (20) and (19) we have

$$e^{i(2\pi n+\xi)}-1=\frac{i\beta_2}{\beta_1-1}\frac{2+O\left(\frac{1}{n}\right)}{2\pi n+O\left(\frac{1}{n}\right)}=\frac{2i\beta_2}{\beta_1-1}\frac{1}{2\pi n}+O\left(\frac{1}{n^2}\right). \tag{22}$$

On the other hand, using Maclaurin expansion of $e^{i\xi}$ and taking into account the second equality of (20) we see that

$$e^{i(2\pi n+\xi)}-1=i\xi+O\left(\frac{1}{n^2}\right)$$

This with (22) gives us (21). Now (16) follows from (20) and (21). Lemma is proved. ■

For $q\neq 0$ it is known that (see (21) of [13]) the characteristic polynomial of $T_1^1(q)$ has the form

$$\Delta(\mu)=\Delta_0(\mu)-\frac{\beta_1+1}{2}\left\{e^{i\mu}(c_\mu-is_\mu)-e^{-i\mu}(c_\mu+is_\mu)\right\}+o\left(\frac{1}{\mu}\right), \tag{23}$$

where $\Delta_0(\mu)$ is defined in (17) and

$$c_\mu=\int_0^1\cos(2\mu t)\,q(t)\,dt,\ \ s_\mu=\int_0^1\sin(2\mu t)\,q(t)\,dt. \tag{24}$$

After some arrangements (23) can be written in the form

$$\Delta(\mu)=\Delta_0(\mu)-\frac{\beta_1+1}{2}e^{-i\mu}\left\{c_\mu\left(e^{2i\mu}-1\right)-is_\mu\left(e^{2i\mu}+1\right)\right\}+o\left(\frac{1}{\mu}\right). \tag{25}$$

Using (17) in this formula we obtain

$$\Delta(\mu) = \left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \beta_2\left(e^{i\mu} + 1\right)\right] -$$
$$-\frac{\beta_1 + 1}{2} e^{-i\mu}\left\{c_\mu\left(e^{2i\mu} - 1\right) - is_\mu\left(e^{2i\mu} + 1\right)\right\} + o\left(\frac{1}{\mu}\right)$$
$$= \left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \beta_2\left(e^{i\mu} + 1\right) - \frac{\beta_1 + 1}{2} c_\mu\left(e^{i\mu} + 1\right)\right] +$$
$$+ i(\beta_1 + 1) s_\mu \cos\mu + o\left(\frac{1}{\mu}\right).$$

Therefore the characteristic determinant $\Delta(\mu)$, can be written as

$$\Delta(\mu) = \Delta_1(\mu) + i(\beta_1 + 1) s_\mu \cos\mu + o\left(\frac{1}{\mu}\right). \tag{26}$$

where

$$\Delta_1(\mu) = \left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right)\left(e^{i\mu} + 1\right)\right]. \tag{27}$$

To obtain the asymptotic formulas for the eigenvalues of $T_1^1(q)$ first let us consider the roots of $\Delta_1(\mu)$.

Lemma 1 *The roots of the function $\Delta_1(\mu)$ consist of the sequences $\left\{\mu_{n,1}^1\right\}$ and $\left\{\mu_{n,2}^1\right\}$ such that*

$$\mu_{n,1}^1 = 2\pi n, \ \ n \in \mathbb{Z}, \tag{28}$$

$$\mu_{n.2}^1 = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + o\left(\frac{1}{n}\right). \tag{29}$$

Proof. The zeros of $\Delta_1(\mu)$ are the zeros of the equations

$$1 - e^{-i\mu} = 0,$$

and

$$g(\mu) =: e^{i\mu} - 1 + \frac{1}{\beta_1 - 1}\left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right)\frac{e^{i\mu} + 1}{i\mu} = 0.$$

The roots of the first equation are $2\pi n$ for $n \in \mathbb{Z}$, that is (28) holds. By definition of $f(\mu)$ (see (19)) we have

$$g(\mu) = f(\mu) - \frac{\frac{\beta_1 + 1}{2} c_\mu}{\beta_1 - 1}\frac{e^{i\mu} + 1}{i\mu}.$$

Since $c_\mu = o(1)$, there exists a sequence δ_n such that $\delta_n = o(1)$ and

$$|g(\mu) - f(\mu)| < \frac{\delta_n}{n} \tag{30}$$

for $\mu \in U(2\pi n)$, where $U(2\pi n)$ is $O\left(\frac{1}{n}\right)$-neighborhood of $2\pi n$.

Now to estimate the zeros of $g(\mu)$, we use Rouche's theorem for the functions $f(\mu)$ and $g(\mu)$ on the circle

$$\gamma_n = \left\{\mu : \mid \mu - \mu_{n,2}(0) \mid = \frac{\varepsilon_n}{n}\right\}, \tag{31}$$

where $\mu_{n,2}(0)$ is defined in (16) and ε_n is chosen so that

$$\varepsilon_n = o(1) \ \& \ \delta_n = o(\varepsilon_n). \tag{32}$$

For this let us estimate $|f(\mu)|$ on γ_n by using the Taylor series of $f(\mu)$ about $\mu_{n,2}(0)$:

$$f(\mu) = f'(\mu_{n,2})(\mu - \mu_{n,2}) + \frac{f''(\mu_{n,2})}{2!}(\mu - \mu_{n,2})^2 + \cdots$$

Since

$$f'(\mu) = ie^{i\mu} - \frac{i\beta_2}{\beta_1 - 1}\frac{ie^{i\mu}}{i\mu} + O\left(\frac{1}{n^2}\right) \sim 1, \ f''(\mu) \sim 1, \ldots,$$

there exist a constant $c > 0$ such that $|f'(\mu)| > c$ and

$$|f(\mu)| > c\frac{\varepsilon_n}{2n} \tag{33}$$

for $\mu \in \gamma_n$. Thus by (30)-(33) and Rouche's theorem, there exists a root $\mu_{n,2}^1$ of $g(\mu)$ inside the circle (31). Therefore (29), follows from (16). ■

Now using (26), (27) and Lemma 1, we get one of the main results of this paper.

Theorem 1 *(a) If (12) holds, then the large eigenvalues of $T_1^1(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\mu_{n,1}(q)\}$ and $\{\mu_{n,2}(q)\}$ satisfying the asymptotic formulas*

$$\mu_{n,1}(q) = 2\pi n + o\left(\frac{1}{n}\right), \tag{34}$$

$$\mu_{n,2}(q) = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + o\left(\frac{1}{n}\right). \tag{35}$$

Moreover the normalized eigenfunctions $\varphi_{n,1}(x)$ and $\varphi_{n,2}(x)$ corresponding to the eigenvalues $(\mu_{n,1}(q))^2$ and $(\mu_{n,2}(q))^2$ satisfy the same asymptotic formula

$$\varphi_{n,j}(x) = \sqrt{2}\cos 2\pi n x + O\left(\frac{1}{n}\right) \tag{36}$$

for $j = 1, 2$

(b) If there exists a subsequence $\{n_k\}$ such that (12) holds whenever n is replaced by n_k, then the system of the root functions of $T_1^1(q)$ does not form a Riesz basis.

Proof. (a) To prove (34) and (35), we show that the large roots of $\Delta(\mu)$ lies in $o\left(\frac{1}{n}\right)$-neighborhood of the roots of $\Delta_1(\mu)$ by using Rouche's theorem for $\Delta(\mu)$ and $\Delta_1(\mu)$ on $\Gamma_1(r_n)$, $\Gamma_2(r_n)$, where

$$\Gamma_j(r_n) = \left\{\mu : \ \left|\mu - \mu_{n,j}^1\right| = r_n\right\}, r_n = o\left(\frac{1}{n}\right) \tag{37}$$

and $\mu_{n,j}^1$ for $j = 1, 2$ are the roots of $\Delta_1(\mu)$. If $\mu \in \Gamma_j(r_n)$ for $j = 1, 2$ then by (12) $s_\mu = o\left(\frac{1}{n}\right)$ and by (26)

$$a(\mu) =: |\Delta(\mu) - \Delta_1(\mu)| < b_n, \ b_n = o\left(\frac{1}{n}\right). \tag{38}$$

We can choose r_n so that

$$b_n = o\left(r_n\right). \tag{39}$$

Now let us estimate $\Delta_1(\mu)$ on the circles $\Gamma_1(r_n)$, $\Gamma_2(r_n)$. By (27)

$$\Delta_1(\mu) = \left(1 - e^{-i\mu}\right) i\mu h(\mu) \tag{40}$$

where

$$h(\mu) = (\beta_1 - 1)\left(e^{i\mu} - 1\right) + \left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right) \frac{e^{i\mu} + 1}{i\mu}. \tag{41}$$

It follows from (28), (29) and (37) that if $\mu \in \Gamma_1(r_n)$ and $\mu \in \Gamma_2(r_n)$ then $\mu = 2\pi n + r_n e^{i\theta}$ and $\mu = 2\pi n + \dfrac{\beta_2}{\beta_1 - 1}\dfrac{1}{\pi n} + r_n e^{i\theta} + o\left(\dfrac{1}{n}\right)$ respectively, where $\theta \in (0, 2\pi)$. Therefore

$$\left(1 - e^{-i\mu}\right) \sim r_n, \tag{42}$$

and

$$\left(1 - e^{-i\mu}\right) \sim \frac{1}{n}, \tag{43}$$

on $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ respectively, where $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$.

Now let us consider $h(\mu)$ on $\Gamma_j(r_n)$, $j = 1, 2$. Since $\mu^1_{n,2}$ is the root of $h(\mu)$ the Taylor expansion of $h(\mu)$ about $\mu^1_{n,2}$ is

$$h(\mu) = h'\left(\mu^1_{n,2}\right)\left(\mu - \mu^1_{n,2}\right) + \frac{h''\left(\mu^1_{n,2}\right)}{2!}\left(\mu - \mu^1_{n,2}\right)^2 + \cdots. \tag{44}$$

By (41), we have

$$h'(\mu) = (\beta_1 - 1) i e^{i\mu} + \left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right) \frac{i e^{i\mu}}{i\mu} + O\left(\frac{1}{n^2}\right) \sim 1$$

for $\mu \in \Gamma_j(r_n)$, $j = 1, 2$. Clearly $h^{(k)}(\mu) \sim 1$ for $k > 1$ and $\mu \in \Gamma_j(r_n)$. On the other hand, $\left(\mu - \mu^1_{n,2}\right) \sim \dfrac{1}{n}$ for $\mu \in \Gamma_1(r_n)$ and $\left(\mu - \mu^1_{n,2}\right) \sim r_n$ for $\mu \in \Gamma_2(r_n)$. Therefore using (44) we obtain

$$h(\mu) \sim \frac{1}{n}, \ \forall \mu \in \Gamma_1(r_n),$$

$$h(\mu) \sim r_n, \ \forall \mu \in \Gamma_2(r_n).$$

These formulas with (40), (42) and (43) imply that

$$\Delta_1(\mu) \sim r_n, \ \forall \mu \in \Gamma_j(r_n) \tag{45}$$

for $j = 1, 2$. Thus by (38), (39), (45) and Rouche's theorem, each of the disks enclosed by the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ contains an eigenvalue which proves (34) and (35).

Since the distance between the centres of the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ is of order $\dfrac{1}{n}$, but $r_n = o\left(\dfrac{1}{n}\right)$, the eigenvalues inside the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ are different, that is, they are simple.

Now let us prove (36). Since the equation

$$-y'' + q(x)y = \mu^2 y$$

has the fundamental solutions of the form

$$y_1(x,\mu) = e^{i\mu x} + O\left(\frac{1}{\mu}\right),\ y_2(x,\mu) = e^{-i\mu x} + O\left(\frac{1}{\mu}\right)$$

(see p. 52 of [18]) the eigenfunctions of $T_1^1(q)$ are

$$y_{n,j}(x) = \begin{vmatrix} e^{i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right) & e^{-i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right) \\ i\mu_{n,j}\left(1+\beta_1 e^{i\mu_{n,j}}\right) + \beta_2 e^{i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right) & -i\mu_{n,j}\left(1+\beta_1 e^{-i\mu_{n,j}}\right) + \beta_2 e^{-i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right) \end{vmatrix}$$

$$= \left[e^{i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right)\right]\left[-i\mu_{n,j}\left(1+\beta_1 e^{-i\mu_{n,j}}\right) + \beta_2 e^{-i\mu_{n,1j}} + O\left(\frac{1}{\mu_{n,j}}\right)\right] -$$

$$- \left[e^{-i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right)\right]\left[i\mu_{n,j}\left(1+\beta_1 e^{i\mu_{n,1}}\right) + \beta_2 e^{i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right)\right].$$

This with the formula

$$\mu_{n,j} = 2\pi n + O\left(\frac{1}{n}\right),$$

for $j = 1,2$ (see (34) and (35)), implies (36).

(b) It is clear that if (12) holds for the subsequence $\{n_k\}$ then (36) holds for $\{n_k\}$ too. Therefore the angle between the eigenfunctions $\varphi_{n_k,1}(x)$ and $\varphi_{n_k,2}(x)$ corresponding to $\mu_{n_k,1}(q)$ and $\mu_{n_k,2}(q)$ tends to zero. Hence the system of the root functions of $T_1^1(q)$ does not form a Riesz basis (see [20]). Note that (b) follows also from (a) and Theorem 2 of [12, 13]. ■

Let q be an absolutely continuous function. Then using the integration by parts formula for s_μ and c_μ defined in (24) we obtain

$$s_\mu = \frac{1}{2\mu}\left[q(0) - q(1)\cos(2\mu)\right] + o(\frac{1}{\mu})$$

and

$$c_\mu = \frac{1}{2\mu} q(1)\sin(2\mu) + o\left(\frac{1}{\mu}\right).$$

If $\mu \in U(2\pi n)$, where $U(2\pi n)$ is defined in the proof of Lemma 1 , then

$$\cos\mu = 1 + O\left(\frac{1}{\mu}\right) \,\&\, \sin\mu = O\left(\frac{1}{\mu}\right)$$

Therefore we have

$$s_\mu = \frac{1}{2\mu}\left[q(0) - q(1)\right] + o(\frac{1}{\mu}),\ c_\mu = o\left(\frac{1}{\mu}\right)$$

and hence by (25)

$$\Delta(\mu) = \Delta_0(\mu) + i(\beta_1+1)s_\mu\cos\mu + o\left(\frac{1}{\mu}\right)$$
$$= \Delta_0(\mu) + \frac{a}{\mu} + o\left(\frac{1}{\mu}\right) \qquad (46)$$

where

$$a = \frac{i(\beta_1+1)}{2}\left[q(0) - q(1)\right].$$

Now we are ready to state the second main result of this paper.

Theorem 2 *Let q be an absolutely continuous function and (13) for $\sigma=1$ hold. Then*

(a) the large eigenvalues of $T_1^1(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\mu_{n,1}(q)\}$ and $\{\mu_{n,2}(q)\}$ satisfying

$$\mu_{n,1}(q)=2\pi n+\frac{2\beta_2-i\sqrt{D}}{4\left(\beta_1-1\right)\pi n}+o\left(\frac{1}{n}\right), \tag{47}$$

$$\mu_{n,2}(q)=2\pi n+\frac{2\beta_2+i\sqrt{D}}{4\left(\beta_1-1\right)\pi n}+o\left(\frac{1}{n}\right). \tag{48}$$

where $D=2\left(1-\beta_1^2\right)\left[q\left(0\right)-q\left(1\right)\right]-\left(2\beta_2\right)^2$

(b) the system of the root functions of $T_1^1(q)$ does not form a Riesz basis.

Proof. (*a*) By (46) $\mu_{n,j}(q)$ is a root of the equation

$$\mu\Delta_0\left(\mu\right)+a+o\left(1\right)=0.$$

Using (17) in this equation we get

$$\mu\left(1-e^{-i\mu}\right)\left[i\mu\left(\beta_1-1\right)\left(e^{i\mu}-1\right)+\beta_2\left(e^{i\mu}+1\right)\right]+a+o\left(1\right)=0. \tag{49}$$

By the Taylor expansions of $e^{-i\mu}$ and $e^{i\mu}$ at $2\pi n$ we have

$$e^{-i\mu}=1-i\left(\mu-2\pi n\right)+O\left(\frac{1}{n^2}\right),$$
$$e^{i\mu}=1+i\left(\mu-2\pi n\right)+O\left(\frac{1}{n^2}\right)$$

for $\mu\in U(2\pi n)$. Therefore (49) can be written in the form

$$i\mu\left(\mu-2\pi n\right)\left[-\mu\left(\beta_1-1\right)\left(\mu-2\pi n\right)+2\beta_2+O\left(\frac{1}{\mu}\right)\right]+a+o\left(1\right)=0. \tag{50}$$

To prove the formulas (47) and (48) we consider the equation (50). In (50) substituting $x=\mu\left(\mu-2\pi n\right)$ and taking into account that $x=O(1)$ for $\mu\in U(2\pi n)$ we get

$$-i\left(\beta_1-1\right)x^2+2i\beta_2x+a+o\left(1\right)=0. \tag{51}$$

To solve (51) we compare the roots of the functions

$$f_1\left(\mu\right)=-i\left(\beta_1-1\right)x^2+2i\beta_2x+a \tag{52}$$

and

$$f_2\left(\mu\right)=-i\left(\beta_1-1\right)x^2+2i\beta_2x+a+\alpha_n \tag{53}$$

on the set $U(2\pi n)$, where $\alpha_n=o\left(1\right)$. The roots of $f_1\left(\mu\right)$ are

$$x_{1,2}=\frac{-2i\beta_2\pm\sqrt{D}}{-2i\left(\beta_1-1\right)} \tag{54}$$

where

$$D=\left(2i\beta_2\right)^2+4i\left(\beta_1-1\right)a=\left(2i\beta_2\right)^2-2\left(\beta_1^2-1\right)\left[q\left(0\right)-q\left(1\right)\right]\neq0. \tag{55}$$

by the assumption (13) for $\sigma = 1$. Therefore we have two different solutions x_1 and x_2.

On the other hand the solutions of the equations $\mu(\mu - 2\pi n) = x_1$ and $\mu(\mu - 2\pi n) = x_2$ with respect to μ are

$$\mu_{11} = O\left(\frac{1}{n}\right), \ \mu_{12} = 2\pi n + \frac{x_1}{2\pi n} + O\left(\frac{1}{n^2}\right)$$

and

$$\mu_{21} = O\left(\frac{1}{n}\right), \ \mu_{22} = 2\pi n + \frac{x_2}{2\pi n} + O\left(\frac{1}{n^2}\right)$$

respectively. Since $x_1 - x_2 \sim 1$ (see (54) and (55)), we have

$$\mu_{12} - \mu_{21} \sim n, \ \mu_{12} - \mu_{22} \sim \frac{1}{n}, \ \mu_{12} - \mu_{11} \sim n. \tag{56}$$

Now consider the roots of $f_2(\mu)$ by using Rouche's theorem on

$$\gamma_j(r_n) = \{\mu : \ |\mu - \mu_{j2}| = r_n\}, \tag{57}$$

for $j = 1, 2$, where r_n is chosen so that

$$r_n = o\left(\frac{1}{n}\right) \ \& \ \alpha_n = o(nr_n). \tag{58}$$

By (52), (53) and (58)

$$|f_1(\mu) - f_2(\mu)| = \alpha_n = o(1)$$

on $\gamma_1(r_n) \cap \gamma_2(r_n)$. Since the roots of $f_1(\mu)$ are μ_{ij} for $i, j = 1, 2$, we have

$$f_1(\mu) = A(\mu - \mu_{11})(\mu - \mu_{12})(\mu - \mu_{21})(\mu - \mu_{22}) \tag{59}$$

where A is a constant. One can easily verify by using (56) and (59) that

$$f'(\mu_{12}) = A(\mu_{12} - \mu_{11})(\mu_{12} - \mu_{21})(\mu_{12} - \mu_{22}) \sim n$$

Since $f(\mu)$ is a polynomial of order 4 we have

$$f''(\mu_{12}) = O(n^2), \ f'''(\mu_{12}) = O(n), \ f^{(4)}(\mu_{12}) = O(1), \ f^{(5)}(\mu_{12}) = 0.$$

Therefore using the Taylor series

$$f_1(\mu) = f_1'(\mu_{12})(\mu - \mu_{12}) + \cdots.$$

of $f_1(\mu)$ about μ_{12} for $\mu \in \gamma_1(r_n)$ and taking into account that $(\mu - \mu_{12}) \sim r_n$ we obtain

$$|f_1(\mu)| \sim nr_n.$$

On the other hand by (58) we have

$$|f_1(\mu) - f_2(\mu)| = \alpha_n = o(nr_n)$$

for $\mu \in \gamma_1(r_n)$. Therefore

$$|f_1(\mu) - f_2(\mu)| < |f_1(\mu)| \tag{60}$$

on $\gamma_1(r_n)$ In the same way we prove that (60) holds on $\gamma_2(r_n)$ too. Hence inside of each of the circles $\gamma_1(r_n)$ and $\gamma_2(r_n)$, there is one root of (49) denoted by $\mu_{n,1}(q)$ and $\mu_{n,2}(q)$ respectively. Since $r_n = o\left(\frac{1}{n}\right)$, $\mu_{n,1}(q)$ and $\mu_{n,2}(q)$ satisfy the formulas (47) and (48). To complete the proof of (a) it is enough to note that disks enclosed by the circles $\gamma_1(r_n)$ and $\gamma_2(r_n)$ have no common points and there are only two roots of (46) in the neighborhood of $2\pi n$. Thus (a) is proved.

(b) The proof of (b) is the same as the proof of Theorem $1(b)$. ■

Now consider $T_1^0(q)$. In this case the characteristic determinant of $T_1^0(0)$ is

$$\Delta_0^0(\mu) = \left(1 + e^{i\mu}\right)\left(i\mu + \beta_1 i\mu e^{-i\mu} - \beta_2 e^{-i\mu}\right) + \left(i\mu + \beta_1 i\mu e^{i\mu} + \beta_2 e^{i\mu}\right)\left(1 + e^{-i\mu}\right) = 0.$$

After simplifying this equation, we have

$$\Delta_0^0(\mu) = \left(1 + e^{-i\mu}\right)\left[i\mu(\beta_1 + 1)\left(e^{i\mu} + 1\right) + \beta_2\left(e^{i\mu} - 1\right)\right] = 0.$$

The roots of this equation has the form

$$(2n+1)\pi, \;\; (2n+1)\pi + \frac{2\beta_2}{\beta_1 + 1}\frac{1}{(2n+1)\pi} + O\left(\frac{1}{n^2}\right).$$

The characteristic determinant of $T_1^0(q)$ can be written in the forms

$$\Delta^0(\mu) = \Delta_0^0(\mu) + \frac{1-\beta_1}{2}e^{-i\mu}\left\{c_\mu\left(e^{2i\mu} - 1\right) - is_\mu\left(e^{2i\mu} + 1\right)\right\} + o\left(\frac{1}{\mu}\right)$$

and

$$\Delta^0(\mu) = \Delta_1^0(\mu) + i(\beta_1 - 1)s_\mu \cos\mu + o\left(\frac{1}{\mu}\right),$$

where

$$\Delta_1^0(\mu) = \left(1 + e^{-i\mu}\right)\left[i\mu(\beta_1 + 1)\left(e^{i\mu} + 1\right) + \left(\beta_2 + \frac{1-\beta_1}{2}c_\mu\right)\left(e^{i\mu} - 1\right)\right].$$

The investigation of $T_1^0(q)$ is similar to the investigation of $T_1^1(q)$. The difference is that, here we consider the functions and equations in $O\left(\frac{1}{n}\right)$-neighborhood of $(2n+1)\pi$ (we denote it by $U((2n+1)\pi)$) instead of $U(2\pi n)$, since the eigenvalues of $T_1^0(0)$ lie in $U((2n+1)\pi)$ while the eigenvalues of $T_1^1(0)$ lie in $U(2\pi n)$. Now instead of Δ_0, Δ_1, Δ using the functions Δ_0^0, Δ_1^0, Δ^0 and repeating the proof of Theorem 1 we obtain:

Theorem 3 *(a) If (12a) holds, then the large eigenvalues of $T_1^0(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\mu_{n,1}^0\}$ and $\{\mu_{n,2}^0\}$ satisfying*

$$\mu_{n,1}^0 = \;\; (2n+1)\pi + o\left(\frac{1}{n}\right),$$

$$\mu_{n,2}^0 = (2n+1)\pi + \frac{2\beta_2}{\beta_1 + 1}\frac{1}{(2n+1)\pi} + o\left(\frac{1}{n}\right).$$

Moreover the normalized eigenfunctions $\varphi_{n,1}^0(x)$ and $\varphi_{n,2}^0(x)$ corresponding to the eigenvalues $\left(\mu_{n,1}^0\right)^2$ and $\left(\mu_{n,2}^0\right)^2$ satisfy the same asymptotic formula

$$\varphi_{n,j}^0(x) = \sqrt{2}\cos(2n+1)\pi x + O\left(\frac{1}{n}\right).$$

for $j=1,2$.

(b) *If there exists a subsequence* $\{n_k\}$ *such that (12a) holds whenever* n *is replaced by* n_k, *then the system of the root functions of* $T_1^0(q)$ *does not form a Riesz basis.*

Now we investigate $T_1^0(q)$, when q is an absolutely continuous function. The analogous formula to (46) is

$$\Delta^0(\mu)=\Delta_0^0(\mu)+\frac{b}{\mu}+o\left(\frac{1}{\mu}\right)=0, \tag{61}$$

where

$$b=\frac{i(1-\beta_1)}{2}\left[q(0)+q(1)\right].$$

Instead of (46) using (61) and repeating the proof of Theorem 2, we obtain:

Theorem 4 *Let* q *be an absolutely continuous function and (13) for* $\sigma=0$ *hold.*

(a) *The large eigenvalues of* $T_1^0(q)$ *are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences* $\{\mu_{n,1}^0\}$ *and* $\{\mu_{n,2}^0\}$ *satisfying*

$$\mu_{n,1}^0=(2n+1)\pi+\frac{2\beta_2-i\sqrt{D_2}}{2(\beta_1+1)(2n+1)\pi}+o\left(\frac{1}{n}\right),$$

$$\mu_{n,2}^0=(2n+1)\pi+\frac{2\beta_2+i\sqrt{D_2}}{2(\beta_1+1)(2n+1)\pi}+o\left(\frac{1}{n}\right),$$

where $D_2=2\left(1-\beta_1^2\right)\left[q(0)+q(1)\right]-\left(2\beta_2\right)^2$.

(b) *The system of the root functions of* $T_1^0(q)$ *does not form a Riesz basis.*

Now we consider $T_2^1(q)$. In this case the characteristic determinant of $T_2^1(0)$ is

$$D_0^1(\mu)=\left(1-e^{i\mu}\right)\left(\beta_3 i\mu+i\mu e^{-i\mu}-\beta_4 e^{-i\mu}\right)+\left(\beta_3 i\mu+i\mu e^{i\mu}+\beta_4 e^{i\mu}\right)\left(1-e^{-i\mu}\right)=0.$$

After simplifying this equation, we have

$$D_0^1(\mu)=\left(1-e^{-i\mu}\right)\left[i\mu(1-\beta_3)\left(e^{i\mu}-1\right)+\beta_4\left(e^{i\mu}+1\right)\right]=0.$$

The roots of this equation has the form

$$2\pi n,\ 2\pi n+\frac{\beta_4}{1-\beta_3}\frac{1}{\pi n}+O\left(\frac{1}{n^2}\right).$$

The characteristic determinant of $T_2^1(q)$ can be written in the forms

$$D^1(\mu)=D_0^1(\mu)-\frac{\beta_3+1}{2}e^{-i\mu}\left\{c_\mu\left(e^{2i\mu}-1\right)-is_\mu\left(e^{2i\mu}+1\right)\right\}+o\left(\frac{1}{\mu}\right)$$

and

$$D^1(\mu)=D_1^1(\mu)+i(\beta_3+1)s_\mu\cos\mu+o\left(\frac{1}{\mu}\right),$$

where

$$D_1^1(\mu)=\left(1-e^{-i\mu}\right)\left[i\mu(1-\beta_3)\left(e^{i\mu}-1\right)+\left(\beta_4-\frac{\beta_3+1}{2}c_\mu\right)\left(e^{i\mu}+1\right)\right].$$

Instead of Δ_0, Δ_1, Δ using the functions D_0^1, D_1^1, D^1 and repeating the proof of Theorem 1 we obtain:

Theorem 5 *(a) If (12) holds, then the large eigenvalues of $T_2^1(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\rho_{n,1}\}$ and $\{\rho_{n,2}\}$ satisfying*

$$\rho_{n,1} = 2\pi n + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2} = 2\pi n + \frac{\beta_4}{1-\beta_3}\frac{1}{\pi n} + o\left(\frac{1}{n}\right).$$

Moreover the normalized eigenfunctions $\phi_{n,1}(x)$ and $\phi_{n,2}(x)$ corresponding to the eigenvalues $(\rho_{n,1})^2$ and $(\rho_{n,2})^2$ satisfy the same asymptotic formula

$$\phi_{n,j}(x) = \sqrt{2}\cos 2\pi n x + O\left(\frac{1}{n}\right)$$

for $j = 1, 2$

(b) If there exists a subsequence $\{n_k\}$ such that (12) holds whenever n is replaced by n_k, then the system of the root functions of $T_2^1(q)$ does not form a Riesz basis.

Let q be an absolutely continuous function. Then analogous formula to (46) is

$$D^1(\mu) = D_0^1(\mu) + \frac{c}{\mu} + o\left(\frac{1}{\mu}\right) = 0, \tag{62}$$

where

$$c = \frac{i(\beta_3+1)}{2}\left[q(0) - q(1)\right].$$

Now instead of (46) using (62) and repeating the proof of Theorem 2, we obtain:

Theorem 6 *Let q be an absolutely continuous function and (14) for $\sigma = 1$ hold. Then*

(a) the large eigenvalues of $T_2^1(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\rho_{n,1}\}$ and $\{\rho_{n,2}\}$ satisfying

$$\rho_{n,1} = 2\pi n + \frac{-2\beta_4 - i\sqrt{D_3}}{4(\beta_3-1)\pi n} + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2} = 2\pi n + \frac{-2\beta_4 + i\sqrt{D_3}}{4(\beta_3-1)\pi n} + o\left(\frac{1}{n}\right),$$

where $D_3 = 2\left(\beta_3^2 - 1\right)\left[q(0) - q(1)\right] - (2\beta_4)^2$.

(b) the system of the root functions of $T_2^1(q)$ does not form a Riesz basis.

Finally, we consider $T_2^0(q)$. In this case the characteristic determinant of $T_2^0(0)$ is

$$D_0^0(\mu) = \left(1 + e^{i\mu}\right)\left(\beta_3 i\mu + i\mu e^{-i\mu} - \beta_4 e^{-i\mu}\right) + \left(\beta_3 i\mu + i\mu e^{i\mu} + \beta_4 e^{i\mu}\right)\left(1 + e^{-i\mu}\right) = 0.$$

After simplifying this equation, we have

$$D_0^0(\mu) = \left(1 + e^{-i\mu}\right)\left[i\mu(1+\beta_3)\left(e^{i\mu}+1\right) + \beta_4\left(e^{i\mu}-1\right)\right] = 0.$$

The roots of this equation has the form

$$(2n+1)\pi,\ \ (2n+1)\pi+\frac{2\beta_4}{\beta_3+1}\frac{1}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right).$$

The characteristic determinant of $T_2^1(q)$ can be written in the forms

$$D^0(\mu)=D_0^0(\mu)+\frac{\beta_3-1}{2}e^{-i\mu}\left\{c_\mu\left(e^{2i\mu}-1\right)-is_\mu\left(e^{2i\mu}+1\right)\right\}+o\left(\frac{1}{\mu}\right)$$

and

$$D^0(\mu)=D_1^0(\mu)+i(1-\beta_3)s_\mu\cos\mu+o\left(\frac{1}{\mu}\right),$$

where

$$D_1^0(\mu)=\left(1+e^{-i\mu}\right)\left[i\mu(1+\beta_3)\left(e^{i\mu}+1\right)+\left(\beta_4+\frac{\beta_3-1}{2}c_\mu\right)\left(e^{i\mu}-1\right)\right].$$

Instead of Δ_0, Δ_1, Δ using the functions D_0^0, D_1^0, D^0 and repeating the proof of Theorem 1 we obtain:

Theorem 7 *(a) If (12a) holds, then the large eigenvalues of $T_2^0(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\rho_{n,1}^0\}$ and $\{\rho_{n,2}^0\}$ satisfying*

$$\rho_{n,1}^0=(2n+1)\pi+o\left(\frac{1}{n}\right),$$

$$\rho_{n,2}^0=(2n+1)\pi+\frac{2\beta_4}{\beta_3+1}\frac{1}{(2n+1)\pi}+o\left(\frac{1}{n}\right).$$

Moreover the normalized eigenfunctions $\phi_{n,1}^0(x)$ and $\phi_{n,2}^0(x)$ corresponding to the eigenvalues $\left(\rho_{n,1}^0\right)^2$ and $\left(\rho_{n,2}^0\right)^2$ satisfy the same asymptotic formula

$$\phi_{n,j}^0(x)=\sqrt{2}\cos(2n+1)\pi x+O\left(\frac{1}{n}\right)$$

for $j=1,2$.

(b) If there exists a subsequence $\{n_k\}$ such that (12a) holds whenever n is replaced by n_k, then the system of the root functions of $T_2^0(q)$ does not form a Riesz basis.

Let q be an absolutely continuous function. Then analogous formula to (46) is

$$D^0(\mu)=D_0^0(\mu)+\frac{d}{\mu}+o\left(\frac{1}{\mu}\right)=0, \tag{63}$$

where

$$d=\frac{i(\beta_3-1)}{2}\left[q(0)+q(1)\right].$$

Now instead of (46) using (63) and repeating the proof of Theorem 2, we obtain:

Theorem 8 *Let q be an absolutely continuous function and (14) for $\sigma=0$ hold. Then*

(a) the large eigenvalues of $T_2^0(q)$ are simple and the square roots (with nonnegative real

part) of these eigenvalues consist of two sequences $\{\rho_{n,1}^{0}\}$ *and* $\{\rho_{n,2}^{0}\}$ *satisfying*

$$\rho_{n,1}^{0}=(2n+1)\,\pi+\frac{2\beta_4-i\sqrt{D_4}}{2\,(\beta_3+1)\,(2n+1)\,\pi}+o\left(\frac{1}{n}\right),$$

$$\rho_{n,2}^{0}=(2n+1)\,\pi+\frac{2\beta_4+i\sqrt{D_4}}{2\,(\beta_3+1)\,(2n+1)\,\pi}+o\left(\frac{1}{n}\right),$$

where $D_4=2\left(\beta_3^2-1\right)[q\,(0)+q\,(1)]-(2\beta_4)^2$.

(*b*) *the system of the root functions of* $T_2^0(q)$ *does not form a Riesz basis.*

References

[1] N. Dernek, O. A. Veliev, On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operators, Israel Journal of Mathematics, 145 (2005) 113-123.

[2] P. Djakov, B. S. Mitjagin, Instability zones of periodic 1-dimensional Schrodinger and Dirac operators, Russian Math. Surveys, 61(4) (2006) 663-776.

[3] P. Djakov, B. S. Mitjagin, Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials, Doklady Mathematics, 83(1) (2011) 5-7.

[4] P. Djakov, B. S. Mitjagin, Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials, Math. Ann. 351(3) (2011) 509–540.

[5] P. Djakov, B. S. Mitjagin, Criteria for existence of Riesz bases consisting of root functions of Hill and 1D Dirac operators, Journal of Functional Analysis, 263(8) (2012) 2300-2332.

[6] N. Dunford, J. T. Schwartz, Linear Operators, Part 3, Spectral Operators, Wiley-Interscience, MR 90g:47001c, New York, 1988.

[7] F. Gesztesy and V. Tkachenko, A Schauder and Riesz Basis Criterion for Non-Self-Adjoint Schrödinger Operators with Periodic and Antiperiodic Boundary Conditions, Journal of Differential Equations, 253 (2012) 400-437.

[8] N. B. Kerimov, Kh. R. Mamedov, On the Riesz basis property of the root functions in certain regular boundary value problems, Math. Notes, 64(4) (1998) 483-487.

[9] G. M. Kesselman, On unconditional convergence of the eigenfunction expansions of some differential operators, Izv. Vuzov, Matematika, 2 (1964) 82-93 (In Russian).

[10] P. Lang, J. Locker, Spectral theory of two-point differential operators determined by -D2, J. Math. Anal. Appl. 146 (1990) 148-191.

[11] A. S. Makin, Convergence of Expansion in the Root Functions of Periodic Boundary Value Problems, Doklady Mathematics, 73(1) (2006) 71-76.

[12] A. S. Makin, On spectral decompositions corresponding to non-self-adjoint Sturm–Liouville operators, Dokl. Math. 73(1) (2006) 15–18.

[13] A. S. Makin, On the basis property of systems of root functions of regular boundary value problems for the Sturm–Liouville operator, Differ. Equ. 42(12) (2006) 1717–1728.

[14] Kh.R. Mamedov, On the basis property in Lp(0; 1) of the root functions of a class non self adjoint Sturm–Liouville operators, Eur. J. Pure Appl. Math. 3(5) (2010) 831–838.

[15] Kh.R. Mamedov, H.Menken, On the basisness in $L_2(0; 1)$ of the root functions in not strongly regular boundary value problems, Eur. J. Pure Appl. Math. 1(2) (2008) 51–60.

[16] H. Menken, Kh.R. Mamedov, Basis property in Lp(0; 1) of the root functions corresponding to a boundary-value problem, J. Appl. Funct. Anal. 5(4) (2010) 351–356.

[17] V. P. Mikhailov, On Riesz bases in $L_2[0, 1]$, Dokl. Akad. Nauk USSR, 114(5) (1962) 981-984.

[18] M. A. Naimark, Linear Differential Operators, George G. Harap&Company, 1967.

[19] C. Nur, O.A. Veliev, On the Basis Property of the Root Functions of Some Class of Non-self-adjoint Sturm-Liouville Operators, arXiv:1301.7043.

[20] A. A. Shkalikov, On the Riesz basis property of the root vectors of ordinary differential operators, Russian Math. Surveys, 34(5) (1979) 249-250.

[21] A. A. Shkalikov, On the basis property of the eigenfunctions of ordinary differential operators with integral boundary conditions, Vestnik Moscow University, Ser. Mat. Mekh. 37(6) (1982) 12-21.

[22] A. A. Shkalikov, O. A. Veliev, On the Riesz basis property of the eigen- and associated functions of periodic and antiperiodic Sturm-Liouville problems, Math. Notes, 85(5) (2009) 647-660.

[23] O. A .Veliev, M. Toppamuk Duman, The spectral expansion for a nonself-adjoint Hill operators with a locally integrable potential, Journal of Math. Analysis and Appl. 265 (2002) 76-90.

[24] O. A. Veliev, On the Nonself-adjoint Ordinary Differential Operators with Periodic Boundary Conditions. Israel Journal of Mathematics, 176 (2010) 195-208.

[25] O. A. Veliev, On the basis property of the root functions of differential operators with matrix coefficients, Central European Journal of Mathematics, 9(3) (2011) 657-672.