Modeling a Supply Chain as a Queuing System

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Abstract: This study investigates a two-stage supply chain consisting of multiple suppliers at the first stage and a manufacturer at the second stage. The suppliers and the manufacturer have limited production capacities. The system operates in a make-to-order environment, i.e. the suppliers and the manufacturer employ make-to-stock and make-to-order strategies, respectively. The inventory of each component at each supplier is controlled via base stock policy. The aim of this study is to model the supply chain as a queuing system. Under the necessary assumptions, each supplier is modeled as an $M / M / 1$ make-to-stock queue. Moreover, the average outstanding backorders and the average inventory level of each supplier are derived using the queuing model. On the other hand, the manufacturer is modeled as a $GI / M / 1$ queue after deriving an approximate distribution for the interarrival times of the manufacturer. Furthermore, the average number of jobs in the manufacturer’s system and the average outstanding backorders at the manufacturer are obtained using the queuing model.

1. INTRODUCTION

A company may operate in different manufacturing environments such as make-to-stock, make-to-order, or manufacture-to-order. In a make-to-stock system, the company reduces the time to fill customer orders by holding inventory. However, besides its competitive advantage, holding inventory also increases the costs and may give rise to other problems associated with keeping stock. If the company operates on a make-to-order basis, no inventory is held and a customer order triggers the manufacturing process. It is expected that in such a system, the delay in filling the customer orders is larger than that of a make-to-stock system. Finally, manufacture-to-order is a hybrid manufacturing strategy, in which the inventories for parts and semi-finished products are held; however, the final products are manufactured after receiving the customer orders. Compared with the other two systems, a manufacture-to-order system leads to lower holding costs than a make-to-stock system and shorter lead times than a make-to-order system.

This paper investigates a two-stage supply chain operating in a manufacture-to-order environment. The system consists of multiple suppliers and a manufacturer with limited production capacities. The suppliers operate on a make-to-stock basis and the manufacturer employs a make-to-order strategy. The manufacturer orders each component from a particular supplier and production cannot start until all components arrive. The inventory of each component at each supplier is controlled via base stock policy. Backorders are allowed in the system and the capacity of the backlog queue at each supplier is infinite. Finally, end-customer demand arrives in single units and it is stochastic.

In this study, the aim is to model the supply chain as a queuing system and to obtain the performance measures of interest, which are the average outstanding backorders and the average inventory level of the suppliers, the average number of jobs in the manufacturer’s system, and the average outstanding backorders at the manufacturer. To our knowledge, the queuing model of a two-stage capacitated supply chain with multiple members at the first stage has not yet been explored in the literature.

This paper is based on two basic assumptions: end-customer demands occur according to a Poisson process; and service times of the suppliers and the manufacturer are independent and exponentially distributed. Under these assumptions, each supplier can be modeled as an $M / M / 1$ make-to-stock queue. On the other hand, to model the manufacturer as a queuing system, the interarrival time distribution of the manufacturer has to be derived first. Then, the manufacturer can be modeled as a $GI / M / 1$ queue based on this distribution.

The paper is organized as follows. Section 2 presents a review of the related literature. In Section 3, an approximate distribution is developed for the interarrival times of the manufacturer. In Section 4, the supply chain is modeled as a queuing system using this approximate distribution and the aforementioned performance measures are obtained. Finally, concluding remarks are given in Section 5.

2. LITERATURE REVIEW

The first model of a series system with stochastic demand was presented in a study by Clark and Scarf (1960). They considered a multi-echelon inventory problem and defined...
they have investigated are \(M/M/1\) and \(GI/G/1\) queues for a system in which backlogging is permitted; and \(M/M/1/Z, M/G/1/Z,\) and \(GI/G/1/Z\) queues for a system with lost sales. They also studied make-to-stock queuing models for systems with stopped arrival, and bulk demand.

Bai et al. (2004) derived the interdeparture time distributions for make-to-stock queues controlled via base stock policy. The main findings of their study are the interdeparture time probability distributions and squared coefficient of variations for the base stock inventory queues with birth and death processes, such as \(M/M/1, M/M/c,\) and \(M/M/\infty\) inventory queues.

Schwarz and Daduna (2006) investigated \(M/M/1\) systems with inventory management. They computed the performance measures and derived the optimality conditions under different inventory management policies such as the \((0,Q)\) policy with and without an additional threshold for the queue length.

Finally, Jouini and Dallery (2008) considered ordinary and conditional first passage times in general birth and death processes. They highlighted that in case of Markovian interarrival and processing times, a make-to-stock system can be modeled as a birth and death process; and some performance measures can be obtained by computing the conditional first passage times. See Lee and Zipkin (1992) and Buzacott and Shanthikumar (1993) for a more detailed review of the literature in this area.

The distinguishing feature of our study is that there are multiple members at the first stage of the supply chain. Therefore, none of the results in the literature can be used to model the manufacturer as a queuing system in our study.

The next section presents the derivation of the interarrival time distribution of the manufacturer, which is needed to approximate the congestion at the manufacturer using a queuing model.

3. INTERARRIVAL TIME DISTRIBUTION OF THE MANUFACTURER

The supply chain considered in this paper has two stages consisting of multiple suppliers and a manufacturer with limited production capacities. Let the number of suppliers be \(n\), where \(n \geq 2\). The suppliers employ a make-to-stock strategy and manage their inventories via base stock policy. Let \(S_i\) be the base stock level of supplier \(i\) for \(i = 1, \ldots, n\). On the other hand, the manufacturer operates on a make-to-order basis, i.e., no inventory is held by the manufacturer.

It is assumed that end-customer demands occur according to a Poisson process with rate \(\lambda\). The service times of supplier \(i\) are independent and identically distributed (i.i.d.) random variables having an exponential distribution with rate \(\mu_i\) for \(i = 1, \ldots, n\). The manufacturer has also i.i.d. and exponentially distributed service times with rate \(\mu_M\). Let \(\rho_i\) and \(\rho_M\) be...
the traffic intensity of supplier \( i \) and the manufacturer, respectively, where traffic intensity is the ratio of the arrival rate to the service rate. For the stability of the system, it is assumed that \( 0 < \rho_i < 1 \) for all \( i = 1, \ldots, n \) and \( 0 < \rho_m < 1 \).

Under these assumptions, each supplier can be modeled as an \( M/M/1 \) make-to-stock queue. On the other hand, the interarrival time distribution of the manufacturer has to be derived to model the manufacturer as a queuing system.

For one supplier, Buzacott et al. (1992) have derived the probability density function \( f_A(t) \) of the interdeparture times of the supplier, i.e. the interarrival times of the manufacturer given by

\[
f_A(t) = \lambda e^{-\lambda t} \left(1 - \rho_j^{S_j} \right) + \mu_j e^{-\mu_j t} \rho_j^{S_j-1} - (\lambda + \mu_j) e^{-(\lambda + \mu_j) t} \left(1 - \rho_j^2 \right) \rho_j^{S_j-1},
\]

(1)

where \( A \) denotes the interarrival time of the manufacturer.

However, as the number of suppliers increases, deriving the distribution of the interarrival times of the manufacturer becomes mathematically intractable. This brings forth the need of an approximate distribution.

Recall that the manufacturer cannot start production until all components arrive. Hence, the supplier with the minimum base stock level is expected to affect the interarrival times of the manufacturer the most. Based on this expectation and according to (1), an appropriate approximation for the probability density function of the interarrival times of the manufacturer can be given by

\[
f_A(t) \approx \lambda e^{-\lambda t} \left(1 - \rho_j^{S_j} \right) + \mu_j e^{-\mu_j t} \rho_j^{S_j-1} - (\lambda + \mu_j) e^{-(\lambda + \mu_j) t} \left(1 - \rho_j^2 \right) \rho_j^{S_j-1},
\]

(2)

where supplier \( j \) is the one with the minimum base stock level among all suppliers, i.e. \( j = \arg \min_{i=1,\ldots,n} S_i \); \( \rho_j \) is the traffic intensity of supplier \( j \); and \( \mu_j \) is the service rate of supplier \( j \).

From (2), the approximate expected value of the interarrival times of the manufacturer is calculated as

\[
E[A] = \frac{1}{\lambda},
\]

(3)

and the approximate variance is

\[
\text{Var}(A) = \frac{1}{\lambda} \left(1 - 2\rho_j^{S_j} \left(1 - \frac{\rho_j}{1 + \rho_j} \right) \right).
\]

(4)

Hence, the approximate squared coefficient of variation, which is the ratio of the variance to the square of the expected value, is given by

\[
CV^2_A = 1 - 2\rho_j^{S_j} \left(1 - \frac{\rho_j}{1 + \rho_j} \right).
\]

(5)

To test the precision of the approximate interarrival time distribution of the manufacturer given in (2), simulation models are developed for two, three, and four suppliers. In each case, the end-customer demand rate is set to one. The traffic intensities of the suppliers and the manufacturer can take on three values: 0.50, 0.67, and 0.80. Three different values are also selected for the base stock levels of the suppliers: 3, 5, and 7. As considering all the combinations is too time consuming, Taguchi designs are created using Minitab 15, each with 27 runs.

The simulation models are developed using Arena 9.0. The replication length of each run is 10,000 time units and the number of replications is set to 10.

To test the precision of the approximate interarrival time distribution of the manufacturer, the Kolmogorov-Smirnov test is applied. The results indicate that at the 0.01 significance level, the approximate distribution fits the interarrival time data of the manufacturer in 79 of the 81 cases, producing an error of 2.47%. However, the exponential distribution fits the data in 23 of the 81 cases, giving an error of 71.60%. Note that the exponential distribution is also considered in this study since it has been used in the literature (Buzacott et al., 1992; Lee and Zipkin, 1992; Gupta and Weerawat, 2006) to approximate the interarrival times of the second stage when there is one member at the first stage. Consequently, since the error for the approximate distribution is reasonable, it is concluded that the interarrival time distribution of the manufacturer can be approximated as given in (2). Using this approximate distribution, the manufacturer can now be modeled as a queuing system. The next section presents the queuing model and performance measures for the suppliers and the manufacturer, respectively.

4. THE QUEUING MODEL AND PERFORMANCE MEASURES

Recall that in our system, end-customer demands occur according to a Poisson process and service times of the suppliers are i.i.d. and exponentially distributed. Under these assumptions, each supplier can be modeled as an \( M/M/1 \) make-to-stock queue. Furthermore, the performance measures of interest are the average outstanding backorders and the average inventory level of each supplier.

Let \( I_i(t) \) be the inventory level of supplier \( i \) at time \( t \); and \( B_i(t) \) be the outstanding backorders at supplier \( i \) at time \( t \) for \( i = 1, \ldots, n \). Then, from Buzacott and Shanthikumar (1993), the average outstanding backorders at supplier \( i \) is calculated as

\[
E[B_i] = \frac{\rho_i^{S_i+1}}{1 - \rho_i}, \quad i = 1, \ldots, n,
\]

(6)

and the average inventory level of supplier \( i \) is given by

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1 If more than one supplier has the minimum base stock level, then supplier \( j \) is the one with the highest traffic intensity among these suppliers.
Then, \( \lambda \) is equal in (3). In addition, since \( C_\lambda \) and \( C_s \) respectively. The authors recommended denotes the number of jobs in the manufacturer’s system and are substituted into (8) is the outstanding backorders at the manufacturer.

For the mean number of jobs in a GI/G/1 queueing system, Shanthikumar and Buzacott (1980) investigated approximations that require only the squared coefficient of variations of the interarrival and service times, denoted by \( C_\lambda^2 \) and \( C_s^2 \), respectively. The authors recommended different approximations for the various values of \( C_\lambda^2 \) and \( C_s^2 \) as given below:

i. The approximation of Krämer and Langenbach-Belz (1976):

\[
E[N_M] = \rho_u + \frac{\rho_u^2 (C_\lambda^2 + C_s^2)}{2(1-\rho_u)} g(C_\lambda^2, C_s^2, \rho_u),
\]

where \( N_M \) denotes the number of jobs in the manufacturer’s system and

\[
g(C_\lambda^2, C_s^2, \rho_u) = \begin{cases} 
\exp \left( \frac{-2(1-\rho_u)(1-C_\lambda^2)}{3\rho_u(C_\lambda^2 + C_s^2)} \right), & C_\lambda^2 \leq 1 \\
\exp \left( \frac{- (1-\rho_u)}{C_\lambda^2 + 4C_s^2} \right), & C_\lambda^2 \geq 1.
\end{cases}
\]

ii. The approximation of Marchal (1976):

\[
E[N_M] = \rho_u + \frac{\rho_u^2 (1+C_s^2)}{1+\rho_u^2 C_s^2} \left( C_\lambda^2 + \rho_u^2 C_s^2 \right) \left( \frac{C_\lambda^2}{2(1-\rho_u)} \right).
\]

iii. The approximation of Page (1972) by adding a slight modification to the original formula:

\[
E[N_M] = \rho_u + \frac{\rho_u^2 (1+C_s^2)}{2(1-\rho_u)} \left( \frac{C_\lambda^2 (1+C_s^2)}{2} + C_s^2 (1-C_\lambda^2) \exp \left( \frac{-2(1-\rho_u)}{3\rho_u} \right) \right).
\]

In addition, there are two other approximations for the mean number of jobs in a GI/G/1 queueing system presented by Buzacott and Shanthikumar (1993). These approximations are given by

\[
E[N_M] = \rho_u + \frac{\rho_u^2 (1+C_s^2)}{2-\rho_u + \rho_u C_s^2} \left( 1+ \rho_u C_s^2 \right) \left( \frac{\rho_u (2-\rho_u) C_\lambda^2 + \rho_u^2 C_s^2}{2(1-\rho_u)} \right)
\]

To select the best-fit approximation for the average number of jobs in the manufacturer’s system among the approximations given in (8)–(12), the simulation models described above are used. Then, the simulation results are compared with these approximations. Note that \( C_\lambda^2 \) is equal to one since the service times of the manufacturer are exponentially distributed. Then, \( C_\lambda^2 \) calculated from (5) and \( C_s^2 = 1 \) are substituted into (8)–(12), and the average numbers of jobs in the manufacturer’s system are calculated accordingly.

The results indicate that in the case of two, three, and four suppliers, the approximation of Marchal (1976) given in (9) has the minimum average errors of 2.74%, 3.28%, and 4.09%, respectively. Since these errors are in acceptable ranges, we select the approximation of Marchal (1976), which leads to the average number of jobs in the manufacturer’s system given by

\[
E[N_M] = \rho_u + \frac{2\rho_u^2}{1+\rho_u^2} \left( \frac{1+\rho_u^2}{1+\rho_u^2} - 2\rho_u^2 \right) \left[ 1-\rho_u \right],
\]

where \( j = \arg \min \rho \). See Fig. 1 for the comparison of the simulation results with the approximation given in (13) for the case of two, three, and four suppliers.

The other performance measure of interest is the average outstanding backorders at the manufacturer, which is equal to the average number of jobs in the manufacturer’s queue since the manufacturer holds no inventory.

Recall that the approximate mean interarrival time of the manufacturer is calculated as \( 1/\lambda \) in (3). In addition, since the service times of the manufacturer are exponentially distributed, the mean service time is given by \( 1/\mu_M \). Then, using Little’s formula, it is easy to prove that

\[
E[N_{sw}] = E[N_M] - \rho_u
\]

where \( N_{sw} \) denotes the number of jobs in the manufacturer’s queue.

Finally, substituting (13) into (14) yields that the average number of jobs in the manufacturer’s queue, i.e. the average outstanding backorders at the manufacturer, can be given by

\[
E[N_{sw}] = E[B_M] \]

\[
= \frac{2\rho_u^2}{1+\rho_u^2} \left( \frac{1+\rho_u^2}{1+\rho_u^2} - 2\rho_u^2 \right) \left[ 1-\rho_u \right],
\]

where \( B_M \) is the outstanding backorders at the manufacturer and \( j = \arg \min \rho \).
Fig. 1. Comparison of the simulation results with the approximation given in (13) for the case of (A) two suppliers; (B) three suppliers; (C) four suppliers.

Fig. 2 depicts the comparison of the simulation results with the approximation given in (15) for the case of two, three, and four suppliers. For each system, the difference between the average number of jobs in the manufacturer’s system given in Fig. 1 and the average outstanding backorders at the manufacturer given in Fig. 2 is approximately equal to the traffic intensity of the manufacturer, as expected from (14).

Consequently, this section presents the queuing model of the supply chain and derivation of the following performance measures: the average outstanding backorders and the average inventory level of the suppliers, the average number of jobs in the manufacturer’s system, and the average outstanding backorders at the manufacturer. The next section continues with concluding remarks.

5. CONCLUDING REMARKS

This paper examines a two-stage supply chain consisting of multiple suppliers and a manufacturer with limited production capacities. The suppliers operate on a make-to-stock basis and apply base stock policy to manage their inventories. On the other hand, the manufacturer employs a make-to-order strategy. The aim of this paper is to model the supply chain as a queuing system. To our knowledge, the queuing model of a two-stage capacitated system with multiple members at the first stage has not yet been explored in the literature.

Under the assumptions of a Poisson demand process and independent and exponentially distributed service times, each supplier is modeled as an $M/M/1$ make-to-stock queue. Furthermore, the average outstanding backorders and the average inventory level of each supplier are derived using the queuing model.

On the other hand, in order to model the manufacturer as a queuing system, an approximate distribution is developed for the interarrival times of the manufacturer. The idea behind the approximation is the expectation that the supplier with the minimum base stock level affects the interarrival times of the manufacturer the most.

To test the precision of the approximate interarrival time distribution of the manufacturer, simulation models are developed for the case of two, three, and four suppliers. The results show that the approximate distribution produces an error of 2.47%, meaning that it can be reasonably used as the interarrival time distribution of the manufacturer. Then, the manufacturer is modeled as a $GI/M/1$ queue. Moreover, the average number of jobs in the manufacturer’s system and the average outstanding backorders at the manufacturer are obtained using the queuing model.

As a further study, one could consider a system in which the manufacturer also operates on a make-to-stock basis. Then, the manufacturer could be modeled as a $GI/M/1$ make-to-stock queue and the new performance measures would be obtained accordingly.
Fig. 2. Comparison of the simulation results with the approximation given in (15) for the case of (A) two suppliers; (B) three suppliers; (C) four suppliers.