ON THE SIMPLICITY OF THE EIGENVALUES OF NON-SELF-ADJOINT
MATHIEU-HILL OPERATORS

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ABSTRACT. Firstly, we analyze some spectral properties of the non-self-adjoint Hill operator
with piecewise continuous even potential. Then using this we find conditions on the potential
of the non-self-adjoint Mathieu operator, such that all eigenvalues of the periodic, antiperiodic,
Dirichlet, and Neumann boundary value problems are simple.

Keywords: Mathieu-Hill Operator, Simple Eigenvalues, Boundary Value Problems.

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1. Introduction and preliminary facts

Let $P(q)$, $A(q)$, $D(q)$, $N(q)$ be the operators in $L_2[0, \pi]$ associated with the equation

$$-y''(x) + q(x)y(x) = \lambda y(x)$$

and the periodic

$$y(\pi) = y(0), \quad y'(\pi) = y'(0),$$

antiperiodic

$$y(\pi) = -y(0), \quad y'(\pi) = -y'(0),$$

Dirichlet

$$y(\pi) = y(0) = 0,$$

Neumann

$$y'(\pi) = y'(0) = 0$$

boundary conditions respectively, where $q$ is a piecewise continuous function. The domains of
definitions of these operators are the set of all functions $f \in L_2[0, \pi]$, satisfying the corresponding
boundary conditions and the conditions $(-f'' + qf) \in L_2[0, \pi]$, $f' \in AC[0, \pi]$, where $AC[0, \pi]$ is
the set of all absolutely continuous functions on $[0, \pi]$.

It is well known that [1], the spectra of the operators $P(q)$ and $A(q)$, consist of the eigenvalues
$\lambda_{2n}$ and $\lambda_{2n+1}$, called periodic and antiperiodic eigenvalues, which are respectively the roots of

$$F(\lambda) = 2 \quad & \quad F(\lambda) = -2,$$

where $n = 0, 1, \ldots$, $F(\lambda) = \varphi'(\pi, \lambda) + \theta(\pi, \lambda)$ is the Hill discriminant, and $\varphi(x, \lambda)$, $\theta(x, \lambda)$ are
the solutions of the equation (1), satisfying the initial conditions

$$\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \quad \theta'(0, \lambda) = \varphi(0, \lambda) = 0.$$

The eigenvalues of the operators $D(q)$ and $N(q)$, called Dirichlet and Neumann eigenvalues, are the
roots of

$$\varphi(\pi, \lambda) = 0 \quad & \quad \theta'(\pi, \lambda) = 0$$

respectively. The spectrum of the operator $L(q)$ associated with (1), and the conditions

$$y(2\pi) = y(0), \quad y'(2\pi) = y'(0)$$

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are the union of the periodic and antiperiodic eigenvalues. In other words, the spectrum of \( L(q) \)
consists of the eigenvalues \( \lambda_n \) for \( n = 0, 1, \ldots \) which are the roots of the equation
\[
(F(\lambda) - 2)(F(\lambda) + 2) = 0.
\]

The operators \( P(q) \), \( A(q) \), \( D(q) \), and \( N(q) \) are denoted respectively by \( P(a, b) \), \( A(a, b) \), \( D(a, b) \),
and \( N(a, b) \), if
\[
q(x) = ae^{-i2x} + be^{i2x},
\]
where \( a \) and \( b \) are complex numbers. If \( b = a \), then, for simplicity of the notations, these
operators are redenoted by \( P(a) \), \( A(a) \), \( D(a) \), and \( N(a) \). The eigenvalues of \( P(a) \) and \( A(a) \) are
denoted by \( \lambda_{2n}(a) \) and \( \lambda_{2n+1}(a) \) for \( n = 0, 1, \ldots \).

We use the following two classical theorems (see p.8-9 of [4] and p.34-35 of [1]).

Theorem 1. If \( q \) is an even piecewise continuous function, then \( \varphi(x, \lambda) \) is an odd function,
and \( \theta(x, \lambda) \) is an even function. Periodic solutions are either \( \varphi(x, \lambda) \) or \( \theta(x, \lambda) \), unless all solutions
are periodic (with period \( \pi \) or \( 2\pi \)). Moreover, the following equality holds
\[
\varphi'(\pi, \lambda) = \theta(\pi, \lambda).
\]

Theorem 2. For all \( n \) and for any nonzero \( a \), the geometric multiplicity of the eigenvalue
\( \lambda_n(a) \) of the operators \( P(a) \) and \( A(a) \) is 1 (that is, there exists one eigenfunction corresponding
to \( \lambda_n(a) \)) and the corresponding eigenfunction is either \( \varphi(x, \lambda_n(a)) \) or \( \theta(x, \lambda_n(a)) \), where, for
simplicity of the notations, the solutions of the equation
\[
-y''(x) + (2a \cos 2x)y(x) = \lambda y(x)
\]
satisfying (7), are denoted also by \( \varphi(x, \lambda) \) and \( \theta(x, \lambda) \).

In [1, 4] these theorems were proved for the real-valued potentials. However, the proofs pass
through for the complex-valued potentials without any change. The spectrum of \( P(a) \), \( A(a) \),
\( D(a) \), \( N(a) \) for \( a = 0 \) are
\[
\{(2k)^2 : k = 0, 1, \ldots\}, \{(2k + 1)^2 : k = 0, 1, \ldots\}, \{k^2 : k = 1, 2, \ldots\}, \{k^2 : k = 0, 1, \ldots\}
\]
respectively. All eigenvalues of \( P(0) \), except 0 and \( A(0) \), are double, while the eigenvalues of
\( D(0) \) and \( N(0) \) are simple. We also use the following result of [7].

Theorem 3. If \( ab = cd \), then the Hill discriminants \( F(\lambda, a, b) \) and \( F(\lambda, c, d) \) (see (6)), for the
operators \( P(a, b) \) and \( P(c, d) \), are the same.

By Theorem 2, the geometric multiplicity of the eigenvalues of \( P(a) \) and \( A(a) \), for any nonzero
complex number \( a \), is \( 1 \). However, in the non-self-adjoint case \( a \in \mathbb{C} \setminus \mathbb{R} \), the multiplicity (algebraic
multiplicity) of these eigenvalues, in general, is not equal to their geometric multiplicity, since the
operators \( P(a) \) and \( A(a) \) may have associated functions (generalized eigenfunctions). Thus,
in the non-self-adjoint case, the multiplicity (algebraic multiplicity) of the eigenvalues may
be any finite number when the geometric multiplicity is 1 (see Chapter 1 of [5]). Therefore
the investigation of the multiplicity of the eigenvalues for complex-valued potential is more
complicated.

In this paper, in Section 2 we first study some spectral properties of the non-self-adjoint Hill
operator with piecewise continuous even potential. In Section 3, we use it to find the conditions
on \( a \), such that all eigenvalues of the operators \( P(a) \), \( A(a) \), \( D(a) \), and \( N(a) \) are simple. Namely
we prove the following

Theorem 4. All eigenvalues of the operators \( A(a) \), \( D(a) \) and \( P(a) \), \( N(a) \) are simple, if
\[
0 < |a| \leq \frac{\sqrt{3}}{9} \quad \text{and} \quad 0 < |a| \leq \frac{1}{3}
\]
respectively.

This theorem with Theorem 3 implies

Theorem 5. All eigenvalues of the operators \( A(a, b) \) and \( P(a, b) \) are simple, if \( 0 < |ab| \leq \frac{32}{9} \)
and \( 0 < |ab| \leq \frac{16}{9} \) respectively.

Note that the estimations of \( ab \) can be improved by using the numerical methods of [2] and
[3].
2. ON THE EVEN POTENTIALS

In this section we analyze, in general, the Hill operator with even piecewise continuous potentials. In the paper [6] the following statements about the connections of the spectra of the operators \( P(q), A(q), D(q), \) and \( N(q) \), where \( q \) is an even potential, were proven.

**Lemma 1 of [6].** If \( \lambda \) is an eigenvalue of both operators \( D(q) \) and \( N(q) \), then

\[
F(\lambda) = \pm 2, \quad \frac{dF}{d\lambda} = 0, \quad (14)
\]

that is, \( \lambda \) is a multiple eigenvalue of \( L(q) \).

**Proposition 1 of [6].** A number \( \lambda \) is an eigenvalue of \( L(q) \), if and only if \( \lambda \) is an eigenvalue of \( D(q) \) or \( N(q) \).

Firstly, using (12) and the Wronskian equalities

\[
\theta(x, \lambda) = \varphi'(x, \lambda) - \varphi(x, \lambda)\theta'(x, \lambda) = 1 \quad (15)
\]

we prove the following improvements of these statements.

**Theorem 6.** Let \( q \) be an even complex-valued function. A complex number \( \lambda \) is both a Neumann and Dirichlet eigenvalue, if and only if it is an eigenvalue of the operator \( L(q) \) with geometric multiplicity 2.

**Proof.** Suppose \( \lambda \) is both a Neumann and Dirichlet eigenvalue, that is, both equality in (8) hold. On the other hand, it follows from (12), (8), and (15) that

\[
\theta(x, \lambda) = \varphi'(x, \lambda) = \pm 1. \quad (16)
\]

Now using (8), (16), and (7), one can easily verify that both \( \theta(x, \lambda) \) and \( \varphi(x, \lambda) \) satisfy either periodic or anti-periodic boundary conditions, that is, \( \lambda \) is an eigenvalue of the operator \( L(q) \) with geometric multiplicity 2.

Conversely, if \( \lambda \) is an eigenvalue of \( L(q) \) with geometric multiplicity 2, then both \( \theta(x, \lambda) \) and \( \varphi(x, \lambda) \) satisfy either periodic or anti-periodic boundary conditions. Therefore by (7), the equalities in (8) hold, that is, \( \lambda \) is both Neumann and Dirichlet eigenvalue.

**Theorem 7.** Let \( q \) be an even complex-valued function. A complex number \( \lambda \) is an eigenvalue of multiplicity \( s \) of the operator \( L(q) \), if and only if it is an eigenvalue of multiplicities \( u \) and \( v \) of the operators \( D(q) \) and \( N(q) \) respectively, where \( u + v = s \) and \( u = 0 \) \((v = 0)\) means that \( \lambda \) is not an eigenvalue of \( D(q) \) \((N(q)) \).

**Proof.** It is well-known and clear that \( \lambda_0 \) is an eigenvalue of multiplicities \( u, v \), and \( s \) of the operators \( D(q), N(q), \) and \( L(q) \) respectively, if and only if

\[
\varphi(x, \lambda) = (\lambda_0 - \lambda)^u f(\lambda), \quad \theta(x, \lambda) = (\lambda_0 - \lambda)^v g(\lambda) \quad (17)
\]

and

\[
(F(\lambda) - 2)(F(\lambda) + 2) = (\lambda_0 - \lambda)^s h(\lambda), \quad (18)
\]

where \( f(\lambda_0) \neq 0, g(\lambda_0) \neq 0, \) and \( h(\lambda_0) \neq 0 \). On the other hand, by (12) and (15), we have

\[
(F(\lambda) - 2)(F(\lambda) + 2) = 4\varphi(x, \lambda) - 4 = 4\varphi(x, \lambda)\varphi'(x, \lambda) - 1 = 4\varphi(x, \lambda)\theta'(x, \lambda). \quad (19)
\]

Thus the proof of the theorem follows from (17)-(19).

To analyze the periodic and antiperiodic eigenvalues in detail, let us introduce the following notations and definitions.

**Definition 1.** Let \( \sigma(T) \) denote the spectrum of the operator \( T \). A number \( \lambda \) is called \( PDN(q) \) (periodic, Dirichlet and Neumann) eigenvalue if \( \lambda \in \sigma(P(q)) \cap \sigma(D(q)) \cap \sigma(N(q)) \). A number \( \lambda \in \sigma(P(q)) \cap \sigma(D(q)) \) is called \( PD(q) \) (periodic and Dirichlet) eigenvalue if it is not \( PDN(q) \) eigenvalue. A number \( \lambda \in \sigma(P(q)) \cap \sigma(N(q)) \) is called \( PN(q) \) (periodic and Neumann) eigenvalue if it is not \( PDN(q) \) eigenvalue. Everywhere replacing \( P(q) \) by \( A(q) \), we get the definition of \( ADN(q) \), \( AD(q) \) and \( AN(q) \) eigenvalues.

Using Theorems 6, 7, Definition 1, and the equality \( \sigma(P(q)) \cap \sigma(A(q)) = \emptyset \), we obtain

**Theorem 8.** Let \( q \) be an even complex-valued function. Then

\[
\theta(x, \lambda) = \varphi'(x, \lambda) = \pm 1. \quad (16)
\]


(a) The spectrum of $P(q)$ is the union of the following three pairwise disjoint sets: \{PDN(q) eigenvalues\}, \{PD(q) eigenvalues\}, and \{PN(q) eigenvalues\}.

(b) A complex number $\lambda$ is an eigenvalue of geometric multiplicity 2 of the operator $P(q)$, if and only if it is $PDN(q)$ eigenvalue.

(c) A complex number $\lambda$ is an eigenvalue of geometric multiplicity 1 of the operator $P(q)$ if and only if it is either $PD(q)$ or $PN(q)$ eigenvalue.

The theorem continues to hold if $P(q)$, $PDN(q), PD(q)$, and $PN(q)$ are replaced by $A(q)$, $ADN(q), AD(q)$, and $AN(q)$ respectively.

Now we prove the main theorem of this section.

**Theorem 9.** Let $q$ be an even complex-valued function, and $\lambda$ be an eigenvalue of geometric multiplicity 1 of the operator $P(q)$. Then the number $\lambda$ is an eigenvalue of multiplicity $s$ of $P(q)$, if and only if it is an eigenvalue of multiplicity $s$ either of the operator $D(q)$ (first case), or of the operator $N(q)$ (second case). In the first case, the system of the root functions of the operators $P(q)$ and $D(q)$ consists of the same eigenfunction $\varphi(x, \lambda)$ and associated functions

$$
\frac{\partial \varphi(x, \lambda)}{\partial \lambda} + \frac{1}{2!} \frac{\partial^2 \varphi(x, \lambda)}{\partial \lambda^2} + \cdots + \frac{1}{(s-1)!} \frac{\partial^{s-1} \varphi(x, \lambda)}{\partial \lambda^{s-1}}.
$$

In the second case, the system of the root function of the operators $P(q)$ and $N(q)$, consists of the same eigenfunction $\theta(x, \lambda)$ and associated functions

$$
\frac{\partial \theta(x, \lambda)}{\partial \lambda} + \frac{1}{2!} \frac{\partial^2 \theta(x, \lambda)}{\partial \lambda^2} + \cdots + \frac{1}{(s-1)!} \frac{\partial^{s-1} \theta(x, \lambda)}{\partial \lambda^{s-1}}.
$$

The theorem continues to hold if $P(q)$ is replaced by $A(q)$.

**Proof.** Let $\lambda$ be an eigenvalue of geometric multiplicity 1, and multiplicity $s$ of the operator $P(q)$. By Theorem 1 there are two cases.

Case 1. The corresponding eigenfunction is $\varphi(x, \lambda)$.

Case 2. The corresponding eigenfunction is $\theta(x, \lambda)$.

We consider Case 1 in the same way one can consider Case 2. In Case 1, $\theta(x, \lambda)$ is not a periodic solution, that is, it does not satisfy the periodic boundary condition (2). On the other hand, the first equality of (6) with (12) and (7) implies that

$$
\theta(\pi, \lambda) = 1 = \theta(0, \lambda),
$$

that is, $\theta(x, \lambda)$ satisfies the first equality in (2). Therefore, $\theta(x, \lambda)$ does not satisfies the second equality of (2), that is,

$$
\theta'(\pi, \lambda) \neq 0.
$$

This inequality means that $v = 0$, where $v$ is defined in Theorem 7. Therefore, by Theorem 7 we have $u = s$, that is, $\lambda$ is an eigenvalue of multiplicity $s$ of the operator $D(q)$.

Now suppose that $\lambda$ is an eigenvalue of multiplicity $s$ of $D(q)$. Then by (8) and (7)

$$
\varphi(\pi, \lambda) = 0 = \varphi(0, \lambda).
$$

On the other hand, using the first equality of (6), (12), and (7) we get

$$
\varphi'(\pi, \lambda) = 1 = \varphi'(0, \lambda).
$$

Therefore, $\varphi(x, \lambda)$ is an eigenfunction of $P(q)$ corresponding to the eigenvalue $\lambda$. Then, by Theorem 1, $\theta(x, \lambda)$ is not a periodic solution. This, as we noted above, implies (23) and the equality $u = s$. Thus, by Theorem 7, $\lambda$ is an eigenvalue of multiplicity $s$ of $P(q)$.

If $\lambda$ is an eigenvalue of multiplicity $s$ of the operators $P(q)$ and $D(q)$, then

$$
F(\lambda) = 2, \quad \frac{dF}{d\lambda} = 0, \quad \frac{d^2F}{d\lambda^2} = 0, \cdots, \frac{d^{s-1}F}{d\lambda^{s-1}} = 0
$$

and

$$
\varphi(\pi, \lambda) = 0, \quad \frac{d\varphi(\pi, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi(\pi, \lambda)}{d\lambda^2} = 0, \cdots, \frac{d^{s-1}\varphi(\pi, \lambda)}{d\lambda^{s-1}} = 0.
$$
Since $\varphi(0, \lambda) = 0$ and $\varphi'(0, \lambda) = 1$ for all $\lambda$, we have
\[
\varphi(0, \lambda) = 0, \quad \frac{d\varphi(0, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi(0, \lambda)}{d\lambda^2} = 0, \ldots, \quad \frac{d^{s-1}\varphi(0, \lambda)}{d\lambda^{s-1}} = 0 \tag{28}
\]
and
\[
\varphi'(0, \lambda) = 1, \quad \frac{d\varphi'(0, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi'(0, \lambda)}{d\lambda^2} = 0, \ldots, \quad \frac{d^{s-1}\varphi'(0, \lambda)}{d\lambda^{s-1}} = 0. \tag{29}
\]
Moreover, using (26) and (12), we obtain
\[
\varphi'(\pi, \lambda) = 1, \quad \frac{d\varphi'(\pi, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi'(\pi, \lambda)}{d\lambda^2} = 0, \ldots, \quad \frac{d^{s-1}\varphi'(\pi, \lambda)}{d\lambda^{s-1}} = 0. \tag{30}
\]
Thus, by (27)-(30), $\varphi(x, \lambda)$ and the functions in (20), satisfy both the periodic and Dirichlet boundary conditions. On the other hand, differentiating $s - 1$ times in $\lambda$ the equation
\[
-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda) \tag{31}
\]
we obtain
\[
-\left(\frac{1}{k!}\frac{\partial^k\varphi(x, \lambda)}{\partial\lambda^k}\right)'' + (q(x) - \lambda)\frac{1}{k!}\frac{\partial^k\varphi(x, \lambda)}{\partial\lambda^k} = \frac{1}{(k - 1)!}\frac{\partial^{k-1}\varphi(x, \lambda)}{\partial\lambda^{k-1}}
\]
for $k = 1, 2, \ldots, (s - 1)$. Therefore $\varphi(x, \lambda)$, and the functions in (20) are the root functions of the operators $P(q)$ and $D(q)$. Thus the first case is proved in the same way we proved the second case. The proofs of these results for $A(q)$ are similar.

3. Main results

In this section, we consider the operators $P(a)$, $A(a)$, $D(a)$, and $N(a)$ with potential
\[
q(x) = 2a\cos 2x, \tag{32}
\]
where $a$ is a nonzero complex number. By Theorem 2, the geometric multiplicity of the eigenvalues of $P(a)$ and $A(a)$ is 1. Therefore, it follows from Theorem 8 that
\[
\sigma(P(a)) = \{PD(a)\text{ eigenvalues}\} \cup \{PN(a)\text{ eigenvalues}\}, \tag{33}
\]
\[
\sigma(A(a)) = \{AD(a)\text{ eigenvalues}\} \cup \{AN(a)\text{ eigenvalues}\}, \tag{34}
\]
where $PD(q)$, $PN(q)$, $AD(q)$, and $AN(q)$ (see Definition 1) are denoted by $PD(a)$, $PD(a)$, $PD(a)$, and $PD(a)$ when the potential $q$ is defined by (32). Moreover, Theorem 7, Theorem 2, and Theorem 9 yield the equalities
\[
\sigma(D(a)) = \{PD(a)\text{ eigenvalues}\} \cup \{AD(a)\text{ eigenvalues}\}, \tag{35}
\]
\[
\sigma(N(a)) = \{PN(a)\text{ eigenvalues}\} \cup \{AN(a)\text{ eigenvalues}\} \tag{36}
\]
and the following theorem.

**Theorem 10.** For any $a \neq 0$ the eigenvalue $\lambda$ of the operator $P(a)$ or $A(a)$ is a multiple, if and only if it is a multiple eigenvalue either of $D(a)$ or $N(a)$. Moreover, the operators $P(a)$, $A(a)$, $D(a)$, and $N(a)$ have associated functions corresponding to any multiple eigenvalues.

Clearly, the eigenfunctions corresponding to $PN(a)$ eigenvalues, $PD(a)$ eigenvalues, $AD(a)$ eigenvalues, and $AN(a)$ eigenvalues have the forms
\[
\Psi_{PN}(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos 2kx, \tag{37}
\]
\[
\Psi_{PD}(x) = \sum_{k=1}^{\infty} b_k \sin 2kx, \tag{38}
\]
\[
\Psi_{AD}(x) = \sum_{k=1}^{\infty} c_k \sin(2k - 1)x, \tag{39}
\]
and

$$\Psi_{AN}(x) = \sum_{k=1}^{\infty} d_k \cos(2k-1)x$$  \hfill (40)

respectively. For simplicity of the calculating, we normalize these eigenfunctions as follows

$$\sum_{k=0}^{\infty} |a_k|^2 = 1, \sum_{k=1}^{\infty} |b_k|^2 = 1, \sum_{k=1}^{\infty} |c_k|^2 = 1, \sum_{k=1}^{\infty} |d_k|^2 = 1.$$  \hfill (41)

Substituting the functions (37)-(40) into (13), we obtain the following equalities

$$\lambda a_0 = \sqrt{2} a_1, (\lambda - 4)a_1 = a\sqrt{2} a_0 + aa_2, (\lambda - (2k)^2)a_k = aa_{k-1} + aa_{k+1},$$  \hfill (42)

$$\lambda - 4)b_1 = ab_2, (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1},$$  \hfill (43)

$$(\lambda - 1)c_1 = ac_1 + ac_2, (\lambda - (2k-1)^2)c_k = ac_{k-1} + ac_{k+1},$$  \hfill (44)

$$(\lambda - 1)d_1 = -ad_1 + ad_2, (\lambda - (2k-1)^2)d_k = ad_{k-1} + ad_{k+1}$$  \hfill (45)

for $k = 2, 3, \ldots$. Here $a_k$, $b_k$, $c_k$, $d_k$ depend on $\lambda$, and $a_0$, $b_1$, $c_1$, $d_1$ are nonzero constants (see [1] p. 34-35).

By Theorem 10, if the eigenvalue $\lambda$ corresponding to one of the eigenfunctions (37)-(40), denoted by $\Psi(x)$, is a multiple, then there exists associated function $\Phi$ satisfying

$$-\left(\Phi(x, \lambda)\right)^\prime + (q(x) - \lambda)\Phi(x, \lambda) = \Psi(x). \hfill (46)$$

Since the boundary conditions (2)-(5) are self-adjoint, the operators of the eigenfunctions satisfy the above 3 equalities: (45), (41) and (47), since these equalities hold if $\lambda$ is a multiple eigenvalue. For this we use the following proposition, which readily follows from (41) and (47).

**Proposition 1.** If there exists $n \in \mathbb{N} = \{1, 2, \ldots\}$ such that

$$|d_n(\lambda)|^2 > \frac{1}{2},$$  \hfill (48)

then $\lambda$ is a simple $AN(a)$ eigenvalue, where $a \neq 0$. The statement continues to hold for $AD(a)$, $PD(a)$, and $PN(a)$ eigenvalues if $d_n$ is replaced by $c_n$, $b_n$, and $a_n$ respectively.

To apply the Proposition 1, we use following lemmas.

**Lemma 1.** Suppose that $\lambda$ is a multiple $AN(a)$ eigenvalue corresponding to the eigenfunction (40), where $a \neq 0$. Then

(a) For all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $k \neq m$, the following inequalities hold

$$|d_k|^2 \leq \frac{1}{2}, \hfill (49)$$

$$|d_k \pm d_m|^2 \leq 1, \hfill (50)$$

$$|d_k|^2 \leq \frac{|a|^2}{|\lambda - (2k-1)^2|^2}. \hfill (51)$$

(b) If $\text{Re} \lambda < (2p - 1)^2 - 2 |a|$ for some $p \in \mathbb{N}$, then $|d_k| > |d_m|$ and

$$|d_{k+s}| < \frac{|2a|^{s+1} |d_{k-1}|}{|\lambda - (2k-1)^2| \ldots |\lambda - (2(k+s) - 1)^2|}.$$  \hfill (52)
for all $k > p$ and $s = 0, 1, ...$

(c) Let $I \subset \mathbb{N}$ and $d(\lambda, I) =: \min_{k \in I} |\lambda - (2k - 1)^2| \neq 0$. Then
\[
\sum_{k \in I} |d_k|^2 \leq \frac{4|a|^2}{(d(\lambda, I))^2}.
\] (53)

(d) If $\lambda$ is a multiple of the $AD(a)$ eigenvalue corresponding to the eigenfunction (39), then the inequalities (49)-(53) continue to hold if $d_j$ is replaced by $c_j$.

**Proof.** (a) If (49) does not hold for some $k$, then by Proposition 1, $\lambda$ is a simple eigenvalue that contradicts the assumption of the lemma. Using the last equalities of (47) and (41), we obtain
\[
|d_k \pm d_m|^2 = \left| - \sum_{n \neq k, m} d_n^2 \pm 2d_kd_m \right| \leq \sum_{n \neq k, m} |d_n|^2 + |d_k|^2 + |d_m|^2 = 1,
\]
that is, (50) holds. Now (51) follows from (45) and (50).

(b) Suppose that $|d_k| \geq |d_{k-1}|$ for some $k > p > 0$. By (45)
\[
|\lambda - (2k - 1)^2| |d_k| \leq |a| |d_{k-1}| + |a| |d_{k+1}|.
\]
On the other hand, using the condition on $\lambda$, we get $|\lambda - (2k - 1)^2| > 2|a|$. Therefore $|d_{k+1}| \geq 2|d_k| - |d_{k-1}| \geq |d_k|$. Repeating this process $s$ times, we obtain $|d_{k+s}| \geq |d_{k+s-1}|$ for all $s \in \mathbb{N}$. But this means that $\{d_{k+s} : s \in \mathbb{N}\}$ is a nondecreasing sequence. On the other hand, $|d_k| + |d_{k+1}| \neq 0$, since if both $d_k$ and $d_{k+1}$ are zero, then using (45) we obtain that $d_j = 0$ for all $j \in \mathbb{N}$, that is, the solutions (40) are identically zero. Therefore, $d_k$ does not converge to zero being the Fourier coefficient of the square integrable function $\Psi_{AN}(x)$. This contradiction shows that $\{d_{k+s} : s \in \mathbb{N}\}$ is a decreasing sequence. Thus $|d_k| > 0$ for all $k > p$.

Now let us prove (52). Using (45) and the inequality $|d_{k-1}| > |d_k| > 0$, we get
\[
|d_{k+s}| < \frac{|2a||d_{k+s-1}|}{|\lambda - (2k + s - 1)|^2},
\] (54)
for all $s = 0, 1, ...$. Iterating (54) $s$ times, we obtain (52).

(c) By (45) we have
\[
\sum_{k \in I} |d_k|^2 \leq \sum_{k \in I} |a|^2 \left( |d_{k-1}| + |d_{k+1}| \right)^2 \leq \sum_{k \in I} \frac{2|a|^2 (|d_{k-1}|^2 + |d_{k+1}|^2)}{(d(\lambda, I))^2}.
\]
Note that in case $k = 1$, instead of $d_{k-1}$, we take $d_1$ (see the first equality of (45)). Now (53) follows from (41).

(d) Everywhere replacing $d_k$ by $c_k$, we get the proof of the last statement.

In a similar way, we prove the following lemma for $P(a)$.

**Lemma 2.** Suppose that $\lambda$ is a multiple $PD(a)$ eigenvalue corresponding to the eigenfunction (38), where $a \neq 0$. Then

(a) For all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, $n \neq m$, the following inequalities hold
\[
|b_m|^2 \leq \frac{1}{2}, \quad |b_n \pm b_m|^2 \leq 1, \quad |b_k|^2 \leq \frac{|a|^2}{|\lambda - (2k)^2|^2}.
\] (55)

(b) If $\text{Re} \lambda < (2p)^2 - 2|a|$ for some $p \in \mathbb{N}$, then $|b_{k-1}| > |b_k| > 0$ and
\[
|b_{k+s}| < \frac{|2a|^{s+1} |b_{k-1}|}{|\lambda - (2k)^2| |\lambda - (2(k + 1))^2| \cdots |\lambda - (2(k + s))^2|},
\] (56)
for all $k > p$ and $s = 0, 1, ...
(c) Let $I \subset \mathbb{N}$ and $b(\lambda, I) = \min_{k \in I} \left| \lambda - (2k)^2 \right| \neq 0$. Then

$$
\sum_{k \in I} |b_k|^2 \leq \frac{4|a|^2}{(b(\lambda,I))^2}.
$$

(57)

(d) If $\lambda$ is a multiple $PN(a)$ eigenvalue corresponding to (37) then the statements (a) and (b) continue to hold for $k > 1$, $m \geq 0$, and the statement (c) continues to hold for $I \subset \{2, 3, \ldots\}$ if $b_j$ is replaced by $a_j$.

Introduce the notation $D_n = \{\lambda \in \mathbb{C} : |\lambda - (2n - 1)^2| \leq 2|a| \}$. 

**Theorem 11.** (a) All eigenvalues of the operator $A(a)$ lie on the unions of $D_n$ for $n \in \mathbb{N}$.

(b) If $4n - 4 > (1 + \sqrt{2})|a|$, where $a \neq 0$, then the eigenvalues of $A(a)$ lying in $D_n$ are simple.

**Proof.** By (34), if $\lambda$ is an eigenvalue of the operator $A(a)$, then the corresponding eigenfunction is $\Psi_{AN}(x)$ or $\Psi_{AD}(x)$ (see (39) or (40)). Without loss of generality, we assume that the corresponding eigenfunction is $\Psi_{AN}(x)$.

(a) Since $d_k \to 0$ as $k \to \infty$, there exists $n \in \mathbb{N}$, such that $|d_n| = \max_{k \in \mathbb{N}} |d_k|$. Therefore (a) follows from (45) for $k = n$.

(b) Suppose that $\lambda \in D_n$ is a multiple eigenvalue corresponding to the eigenfunction $\Psi_{AN}(x)$. By definition of $D_n$ for $k \neq n$ we have

$$
|\lambda - (2k - 1)^2| \geq |(2n - 1)^2 - (2k - 1)^2| - 2|a| \geq |(2n - 3)^2 - (2n - 1)^2| - 2|a|.
$$

This, together with the condition on $n$ and the definition of $d(\lambda, I)$ (see Lemma 1(c)), gives $d(\lambda, \mathbb{N}\{n\}) > 2\sqrt{2}|a|$. Thus, using (53) and (41), we get

$$
\sum_{k \neq n} |d_k|^2 < \frac{1}{2} \quad \& \quad |d_n|^2 > \frac{1}{2}
$$

which contradicts Proposition 1.

Instead of Lemma 1, using Lemma 2 in the same way, we prove the following:

**Theorem 12.** (a) All $PD(a)$ eigenvalues lie in the unions of $B = \{\lambda : |\lambda - 4| \leq |a| \}$ and $B_n = : \{\lambda : |\lambda - (2n^2)| \leq 2|a| \}$ for $n = 2, 3, \ldots$. All $PN(a)$ eigenvalues lie in the unions of $A_0 = \{\lambda : |\lambda| \leq 2|a| \}$, $A_1 = \{\lambda : |\lambda - 4| \leq (1 + \sqrt{2})|a| \}$ and $B_n$ for $n = 2, 3, \ldots$.

(b) If $4n - 2 > (1 + \sqrt{2})|a|$ and $n > 1$, where $a \neq 0$, then the eigenvalues of $P(a)$ lying in $B_n$ are simple.

Now we prove the main result for $A(a)$.

**Theorem 13.** If $0 < |a| \leq \frac{2}{\sqrt{6}}$, then all eigenvalues of the operator $A(a)$ are simple.

**Proof.** Since $8 > \frac{2}{\sqrt{6}}(1 + \sqrt{2})$, by Theorem 11(b), the ball $D_n$ for $n > 2$ does not contain the multiple eigenvalues of the operator $A(a)$. Therefore, we need to prove that the ball $D_n$ for $n = 1, 2$ also does not contain the multiple eigenvalues. Since the balls $D_1$ and $D_2$ are contained in the half plane $\{\lambda \in \mathbb{C} : \text{Re} \lambda < 16 \}$, we consider the following two strips $\{\lambda \in \mathbb{C} : 9 < \text{Re} \lambda < 16 \}$, $\{\lambda \in \mathbb{C} : 6 < \text{Re} \lambda \leq 9 \}$ and half plane $\{\lambda \in \mathbb{C} : \text{Re} \lambda \leq 6 \}$ separately. We consider the $AN(a)$ eigenvalues, that is, the eigenvalues corresponding to the eigenfunctions (40). Consideration of the $AD(a)$ eigenvalues are the same.

To prove the simplicity of the eigenvalues lying in the above strips, we assume that $\lambda$ is a multiple eigenvalue. Using Lemma 1 by direct calculate (see Estimation 1 and Estimation 2 in Appendix), we show that (48) for $n = 2$ holds, which contradicts Proposition 1.

Investigating the half plane $\text{Re} \lambda \leq 6$ is more complicated. Here we use the first two equalities of (45)

$$
(\lambda - 1)d_1 = -ad_1 + ad_2, \quad (\lambda - 9)d_2, = ad_1 + ad_3.
$$

(58)

By direct calculating, we get (see Estimation 3 and Estimation 4 in the Appendix)

$$
\sum_{k=3}^{\infty} |d_k|^2 < 0.03415, \quad \left| \frac{d_3}{d_2} \right| < 0.17432.
$$

(59)
Then by (41) we have
\[ |d_1|^2 + |d_2|^2 > 1 - \varepsilon, \]
where \( \varepsilon = 0.03415 \). On the other hand, by (49), \( |d_1|^2 \leq \frac{1}{2}, |d_2|^2 \leq \frac{1}{2} \). These inequalities and (47) imply that
\[ |d_1|^2 = \frac{1}{2} - \varepsilon_1, |d_2|^2 = \frac{1}{2} - \varepsilon_2, d_2^2 = -d_1^2 + \varepsilon_3, \]
where \( \varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon, |\varepsilon_3| < 0.03415 \). Now, one can easily see that
\[ \left( \frac{d_2}{d_1} \right)^2 = -1 + \alpha, \frac{d_2}{d_1} = \pm(i + \delta), \]
where \( |\alpha| < \frac{0.03415}{0.9 - 0.03415} < 0.074, |\delta| < \frac{1}{2} |0.074| + \frac{1}{9} |0.074|^2 < 0.04 \). Therefore we have
\[ \frac{d_2}{d_1} - \frac{d_1}{d_2} = \frac{2i(i + \delta)}{i + \delta} = \pm \frac{2i(i + \delta)}{i + \delta} = \pm 2i + \gamma, \]
where \( |\gamma| < \frac{(0.04)^2}{1 - 0.04} < 0.002 \). On the other hand, dividing the first equality of (58) by \( d_1 \), and the second by \( d_2 \), and then subtracting second from the first and using (61), we get
\[ \frac{8}{\alpha} = \pm 2i - 1 + \gamma - \frac{d_3}{d_2}, \]
where by assumption \( \left| \frac{8}{\alpha} \right| \geq \sqrt{6} \). Therefore, using the second estimation of (59) in (62), we get the contradiction \( 2.4495 < \sqrt{6} \leq \left| \frac{8}{\alpha} \right| < \sqrt{5} + 0.17432 + 0.002 < 2.4125 \).

In the same way we consider the simplicity of the eigenvalues of the operators \( P(a), D(a) \), and \( N(a) \). First let us investigate the eigenvalues of \( D(a) \). Since the eigenvalues of \( D(a) \) are the union of \( PD(a) \) and \( AD(a) \) eigenvalues, and the \( AD(a) \) eigenvalues are investigated in Theorem 13, we investigate the \( PD(a) \) eigenvalues.

**Theorem 14.** If \( 0 < |a| \leq 5 \), then all \( PD(a) \) eigenvalues are simple. Moreover, if \( 0 < |a| \leq \frac{8}{\sqrt{6}} \), then all eigenvalues of the operator \( D(a) \) are simple.

**Proof.** The second statement follows from the first statement and Theorem 13. Therefore we need to prove the first statement by using (43). Since \( 14 > 5(1 + \sqrt{2}) \), by Theorem 12, the \( PD(a) \) eigenvalues lying in the ball \( B_n \) for \( n > 3 \) are simple.

If \( \lambda \in B_3 \), then \( 26 \leq \text{Re} \lambda \leq 46 \). Using Lemma 2 and (41), we obtain the estimations (see Estimation 5 in Appendix)
\[ \sum_{k \neq 3} |b_k|^2 < \frac{1}{2}, |b_3|^2 > \frac{1}{2}, \]
which, by Proposition 1, proves the simplicity of the \( PD(a) \) eigenvalues lying in \( B_3 \).

Now we need to prove that the balls \( B \) and \( B_2 \) do not contain the multiple \( PD(a) \) eigenvalues. Since these balls are contained in the strip \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq 26 \} \), we consider the following cases: \( 16 < \text{Re} \lambda \leq 26, 12 < \text{Re} \lambda \leq 16 \) and \( \text{Re} \lambda \leq 12 \).

In the first two cases, using Lemma 2, we get the inequality (see Estimation 6 and Estimation 7) obtained from (48) for \( n = 2 \), by replacing \( d_n \) with \( b_n \), which proves, by Proposition 1, the simplicity of the eigenvalues. Now consider the third case \( \text{Re} \lambda \leq 12 \). Using Lemma 2, we obtain (see Estimation 8 and Estimation 9 in Appendix)
\[ \sum_{k=3}^{\infty} |b_k|^2 < \frac{1}{15}, \frac{|b_3|}{|b_2|} < 0.2131. \]

The first inequality of (63) with (41) implies that
\[ |b_1|^2 + |b_2|^2 > 1 - \beta, \]
where \( \beta < \frac{1}{15} \). Instead of (60), using (64), and repeating the proof of (61), we obtain

\[
\frac{b_2 - b_1}{b_1} = \frac{(i + \delta)^2 - 1}{i + \delta} = \frac{2i(i + \delta) + \delta^2}{i + \delta} = \pm 2i + \gamma_1,
\]

where \(|\gamma_1| < 0.01\). Now dividing the first equality of (43) by \(b_1\), and the second equality of (43) for \(k = 2\) by \(b_2\), and then subtracting the second from the first, and using (65), we get

\[
\frac{12}{a} = \pm 2i + \gamma_1 - \frac{b_1}{b_2},
\]

where by assumption \(\left|\frac{12}{a}\right| \geq 2.4\). Thus, using (63) in (66), we get the contradiction \(2.4 \leq \left|\frac{12}{a}\right| < 2 + 0.2131 + 0.01 = 2.2231\).

**Theorem 15.** If \(0 < |a| \leq \frac{4}{3}\), then all eigenvalues of \(P(a)\) and \(N(a)\) are simple.

**Proof.** By Theorem 13 and Theorem 14, we need to prove that if \(|a| \leq \frac{4}{3}\), then all \(PN(a)\) eigenvalues are simple. Since \(6 > (1 + \sqrt{2})\frac{4}{3}\), by Theorem 12, the \(PN(a)\) eigenvalues lying in the ball \(B_n\) for \(n > 1\) are simple.

Now we prove that the balls \(A_0\) and \(A_1\) do not contain the multiple \(PN(a)\) eigenvalues. Since these balls are contained in \(\{\lambda \in \mathbb{C} : \text{Re} \lambda < 8\}\), we consider the following cases:

**Case 1:** \(3 \leq \text{Re} \lambda < 8\). Using (42) and Lemma 2, (see Estimation 10 in Appendix), we obtain \(|a_1|^2 > \frac{1}{3}\), which by Proposition 1, proves the simplicity of the eigenvalues.

**Case 2:** \(\text{Re} \lambda < 3\). Using Lemma 2, we obtain (see Estimations 11 and 12 in Appendix)

\[
\sum_{k=2}^{\infty} |a_k|^2 < \frac{1}{58}, \quad \frac{|a_2|}{|a_1|} < 0.10301.
\]

The first inequality of (67) with (41) implies that

\[
|a_0|^2 + |a_1|^2 > 1 - \rho,
\]

where \(\rho < \frac{1}{58}\). Instead of (60), using (68) and repeating the proof of (61), we obtain

\[
\frac{a_1}{a_0} - \frac{a_0}{a_1} = \pm 2i + \gamma,
\]

where \(|\gamma| < 0.0006\). Now dividing the first equality of (42) by \(a_0\), and the second by \(a_1\), and then subtracting the second from the first, and taking into account (69), we get

\[
\frac{4}{a} = \pm 2\sqrt{2i + \sqrt{2}} - \frac{a_2}{a_1},
\]

where by assumption \(\left|\frac{4}{a}\right| \geq 3\). Therefore, using (67) we get the contradiction \(3 \leq \left|\frac{4}{a}\right| < \sqrt{2}(2 + 0.0006) + 0.10301 = 2.9323\).

4. Appendix

**Estimation 1:** Let \(9 < \text{Re} \lambda < 16\) and \(|a| \leq \frac{8}{\sqrt{6}}\). Using (51), (53), and taking into account that \(d(\lambda, \{4, 5, \ldots\}) < 33\), we get

\[
|d_1|^2 \leq \frac{|a|^2}{|\lambda - 1|^2} \leq \frac{1}{6}, \quad |d_3|^2 \leq \frac{|a|^2}{|\lambda - 25|^2} \leq \frac{32}{243}, \quad \sum_{k=4}^{\infty} |d_k|^2 < \frac{128}{3267}, \quad \sum_{k \neq 2} |d_k|^2 < \frac{1}{2}.
\]

**Estimation 2.** Let \(6 < \text{Re} \lambda \leq 9\) and \(|a| \leq \frac{8}{\sqrt{6}}\). By (51), (53), and by \(d(\lambda, \{4, 5, \ldots\}) \leq 40\) we have

\[
|d_1|^2 \leq \frac{32}{73}, \quad |d_3|^2 \leq \frac{1}{24}, \quad \sum_{k=4}^{\infty} |d_k|^2 \leq \frac{2}{73}, \quad \sum_{k \neq 2} |d_k|^2 < \frac{1}{2}.
\]
Estimation 3. Let \( \text{Re} \lambda \leq 6 \) and \( |a| \leq \frac{8}{\sqrt{6}} \). By (52) and (49), we have

\[
|d_4| \leq \frac{2 \times \frac{8}{\sqrt{6}}}{|43| |19|} |d_2| \leq \frac{2 \times \frac{8}{\sqrt{6}}}{|43| |19|} \frac{\sqrt{2}}{2}, \quad |d_5| \leq \frac{2 \times \frac{8}{\sqrt{6}}}{|75| |43| |19|} |d_2| \leq \frac{2 \times \frac{8}{\sqrt{6}}}{|75| |43| |19|} \frac{\sqrt{2}}{2}. \tag{71}
\]

Now using (51) and (53), and taking into account \( d(\lambda, \{6, 7, \ldots\}) \leq 115 \), we obtain

\[
|d_3|^2 \leq \frac{8}{|19|} = \frac{32}{1083} \quad \text{and} \quad \sum_{k=6}^{\infty} |d_k|^2 \leq \frac{4}{|115|^2}, \quad \sum_{k=3}^{\infty} |d_k|^2 < 0.03415.
\]

Estimation 4. Now we estimate \( \frac{|d_1|}{|d_2|} \) for \( \text{Re} \lambda \leq 6 \) and \( |a| \leq \frac{8}{\sqrt{6}} \). Iterating (45) for \( k = 3 \), we get

\[
d_3 = \frac{a d_2 + a d_4}{\lambda - 25} = \frac{a d_2}{\lambda - 25} + \frac{a (a d_3 + a d_5)}{\lambda - 25} \tag{72}
\]

\[
= \frac{a d_2}{\lambda - 25} + \frac{a^3 d_2}{(\lambda - 25)(\lambda - 49)} + \frac{a^3 d_4}{(\lambda - 25)^2(\lambda - 49)} + \frac{a^2 d_5}{(\lambda - 25)(\lambda - 49)}. \]

Therefore, dividing both sides of (72) by \( d_2 \), and using (52), we obtain

\[
\frac{|d_3|}{|d_2|} \leq \frac{\frac{8}{\sqrt{6}}}{19} + \frac{\frac{8}{\sqrt{6}}}{|43| |19|^2} + \frac{\frac{4}{|115|^2}}{|43|^2 |19|^3} + \frac{\frac{8}{|75|^2 |43|^2 |19|^2}}{|115|^2} \leq 0.17432.
\]

Estimation 5. Let \( 26 \leq \text{Re} \lambda \leq 46 \) and \( |a| \leq 5 \). Using (56) and (58), we get

\[
|b_1|^2 \leq \frac{|a|^2}{|\lambda - 4|^2} \leq \frac{|5|^2}{|22|^2} = \frac{25}{484}, \quad |b_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|5|^2}{|10|^2} = \frac{1}{4},
\]

\[
|b_3|^2 \leq \frac{|a|^2}{|\lambda - 64|^2} \leq \frac{|5|^2}{|18|^2} = \frac{25}{324}, \quad \sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|54|^2} = \frac{25}{729} \sum_{k=3}^{\infty} |b_k|^2 < \frac{1}{2}.
\]

Estimation 6. Let \( 16 < \text{Re} \lambda \leq 26 \) and \( |a| \leq 5 \). By (55) and (57), we have

\[
|b_1|^2 \leq \frac{|a|^2}{|\lambda - 4|^2} \leq \frac{|5|^2}{|12|^2} = \frac{25}{144}, \quad |b_3|^2 \leq \frac{|a|^2}{|\lambda - 36|^2} \leq \frac{|5|^2}{|10|^2} = \frac{1}{4},
\]

\[
\sum_{k=4}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|38|^2} = \frac{25}{361}, \quad \sum_{k=2}^{\infty} |b_k|^2 \leq \frac{25}{144} + \frac{1}{4} + \frac{25}{361} = \frac{25621}{51984} < \frac{1}{2}.
\]

Estimation 7. Let \( 12 < \text{Re} \lambda \leq 16 \) and \( |a| \leq 5 \). By (55) and (57), we have

\[
|b_1|^2 \leq \frac{|5|^2}{|8|^2} = \frac{25}{64}, \quad |b_3|^2 \leq \frac{|5|^2}{|20|^2} = \frac{1}{16}, \quad |b_4|^2 \leq \frac{|5|^2}{|48|^2} = \frac{25}{2304},
\]

\[
\sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|84|^2} = \frac{25}{1764}, \quad \sum_{k=2}^{\infty} |b_k|^2 \leq \frac{25}{64} + \frac{1}{16} + \frac{25}{2304} + \frac{25}{1764} = \frac{53981}{112896} < \frac{1}{2}.
\]

Estimation 8. Let \( \text{Re} \lambda \leq 12 \) and \( |a| \leq 5 \). Using (55) and (57), we obtain

\[
|b_1|^2 \leq \frac{|5|^2}{|52|^2} = \frac{25}{2704}, \quad |b_3|^2 \leq \frac{|5|^2}{|24|^2} = \frac{25}{576},
\]

\[
\sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|88|^2} = \frac{25}{1936}, \quad \sum_{k=3}^{\infty} |b_k|^2 \leq \frac{25}{2704} + \frac{25}{576} + \frac{25}{1935} = \frac{30495}{465088} < \frac{1}{15}.
\]
Estimation 9. Here we estimate \( |a_1|/|b_2| \) for Re \( \lambda \leq 12 \) and \( |a| \leq 5 \). Iterating (43) for \( k = 3 \), we get

\[
b_3 = \frac{ab_2 + ab_4}{\lambda - 36} = \frac{ab_2}{\lambda - 36} + \frac{a}{\lambda - 36} (\frac{ab_3 + ab_5}{\lambda - 64}) =
\]

Now dividing both sides of (73) by \( b_2 \), and using (56) we obtain

\[
\frac{|b_3|}{|b_2|} \leq \frac{5}{24} + \frac{|5|^3}{|52||24|^2} + \frac{4|5|^5}{|52|^2|24|^3} + \frac{8|5|^5}{|88||52|^2|24|^2} < 0.2131.
\]

Estimation 10. Let \( 3 \leq \text{Re} \lambda < 8 \) and \( |a| \leq \frac{4}{3} \). By (42), Lemma 2(d), and (55), we have

\[
|a_0|^2 \leq \frac{\sqrt{2}a a_1^2}{|\lambda|^2} \leq \frac{|\lambda|^2}{|\lambda|^2} = \frac{16}{81}, \quad |a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|a|^2}{8^2} = \frac{1}{36}.
\]

\[
\sum_{k=3}^{\infty} |a_k|^2 \leq \frac{4|\lambda|^2}{28^2} = \frac{441}{28^2}, \quad \sum_{k=3}^{\infty} |a_k|^2 \leq \frac{16}{81} + \frac{1}{36} + \frac{4}{441} < \frac{1}{2}.
\]

Estimation 11. Let Re \( \lambda < 3 \) and \( |a| \leq \frac{4}{3} \). By Lemma 2(d), (55), and (57), we have

\[
|a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{16}{1561}, \quad \sum_{k=3}^{\infty} |a_k|^2 \leq \frac{|\lambda|^2}{|33|^2} = \frac{64}{9801}, \quad \sum_{k=2}^{\infty} |a_k|^2 < \frac{1}{58}.
\]

Estimation 12. Here we estimate \( a_{21} \) for Re \( \lambda < 3 \) and \( |a| \leq \frac{4}{3} \). Iterating (42) for \( k = 2 \), we get

\[
a_2 = \frac{a a_1 + a_3}{\lambda - 16} = \frac{a a_1}{\lambda - 16} + \frac{a}{\lambda - 16} (\frac{a a_2 + a_4}{\lambda - 36}) =
\]

Now dividing both sides of (74) by \( a_1 \), and using Lemma 2(d) and (56), we obtain

\[
\left| \frac{a_2}{a_1} \right| \leq \frac{4}{13} + \frac{|4|^3}{|33||13|^2} + \frac{|4|^5}{|33|^2|13|^2} + \frac{|8|^5}{|61||33|^2|13|^2} < 0.10301.
\]

References

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