On multiplication lattice modules

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Abstract

In this paper we study multiplication lattice modules. Next we characterize hollow lattices modules. We also establish maximal elements in multiplication lattices modules.In [16], we introduced the concept of a multiplication lattice $L$-module and we characterized it by principal elements.In this paper, we continue study on multiplication lattice $L$-module.

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1.

A multiplicative lattice $L$ is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element $1_L$ (least element $0_L$) as a multiplicative identity (zero). Multiplicative lattices have been studied extensively by E.W.Johnson, C.Jayaram, the current authors, and others, see, for example, [1 – 16].

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1_L$ in $L$ is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in $L$ is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. It is easily seen that maximal elements are prime.

If $a, b$ belong to $L$, $(a \land b)$ is the join of all $c \in L$ such that $cb \leq a$. An element $e$ of $L$ is called meet principal if $a \land e = ((a \land e) \land b) e$ for all $a, b \in L$. An element $e$ is called join principal if $(ae \lor b : L e) = a \lor (b : L e)$ for all $a, b \in L$. $e \in L$ is said to be principal if $e$ is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \land e = e$ for all $a \in L$. $e \in L$ is called compact if $a = \lor b_0$ implies $a \leq b_0 \lor b_{\alpha_2} \lor \ldots \lor b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If each element of $L$ is a join of principal (compact) elements of $L$, then $L$ is called a PG−lattice (CG − lattice).

Let $M$ be a complete lattice. Recall that $M$ is a lattice module over the multiplicative lattice $L$, or simply an $L$-module in the case there is a multiplication between elements of $L$ and $M$, denoted by $lB$ for $l \in L$ and $B \in M$, which satisfies the following properties :
(i) \( (lb) B = l(bB) \);
(ii) \( \left( \bigvee l_\alpha \right) \left( \bigvee B_\beta \right) = \bigvee l_\alpha B_\beta \);
(iii) \( 1LB = B \);
(iv) \( 0LB = 0M \) for all \( l, l_\alpha, b \) in \( L \) and for all \( B, B_\beta \) in \( M \).

Let \( M \) be an \( L \)-module. If \( N, K \) belong to \( M \), \( (N :_L K) \) is the join of all \( a \in L \) such that \( aK \leq N \). If \( a \in L \), \( (0_L : a M) \) is the join of all \( H \in M \) such that \( aH = 0_M \). An element \( N \) of \( M \) is called meet principal if \( (bN : b M) N = bN \cap B \) for all \( b \in L \) and for all \( B \in M \). An element \( N \) of \( M \) is called join principal if \( b \cap (B : M) N = ((bN \cap B) : M) N \) for all \( b \in L \) and for all \( N \in M \). \( N \) is said to be principal if it is both meet principal and join principal. In special case an element \( N \) of \( M \) is called weak meet principal (weak join principal) if \( (bN : M) N = bN \cap M \) \((bN : M) N = bM \cap B \) for all \( b \in M \) and for all \( b \in L \). \( N \) is called to be weak principal if \( N \) is both weak meet principal and weak join principal.

Let \( M \) be an \( L \)-module. An element \( N \in M \) is called compact if \( N \leq \bigvee B_\alpha \) implies \( N \leq B_{\alpha_1} \sqcup B_{\alpha_2} \sqcup \ldots \sqcup B_{\alpha_n} \) for some subset \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \). The greatest element of \( M \) will be denoted by \( 1_M \). If each element of \( M \) is a join of principal (compact) elements of \( M \), then \( M \) is called a \( PG \)-lattice (CG-lattice).

Let \( L \) be a multiplicative lattice and let \( M \) be an \( L \)-module. If \( M \) is \( CG \)-lattice, then any weak principal element \( N \) of \( M \) is compact [14, Corollary 2.2]. Especially, if \( L \) is a \( CG \)-lattice, then any weak principal element in \( L \) is compact [14, Corollary 2.3].

For various characterizations of lattice modules, the reader is referred to [9 – 16].

H.M. Nakkar and I.A. Al-Khouja [13, 14] studied multiplicative lattice modules over multiplicative lattices. In [16], we introduced the concept of a multiplication lattice module and we characterized it by principal elements. In this study, we continue study on multiplicative lattice \( L \)-module and we prove that many important theorems like Nakayama Lemma. We also prove that if \( L \) is a multiplicative \( PG \)-lattice and \( M \) is a multiplication \( PG \)-lattice module, then \( K \) is maximal element of \( M \) if and only if there exist a maximal element \( p \in L \) such that \( K = p1_M < 1_M \).

1.1. Definition. Let \( L \) be a multiplicative lattice and \( c \in L \). \( c \) is said to be a multiplication element if for every element \( a \in L \) such that \( a \leq c \) there exists an element \( d \in L \) such that \( a = cd \).

1.2. Definition. Let \( L \) be a multiplicative lattice and \( M \) a lattice \( L \)-module. \( N \in M \) is said to be a multiplication element if for every element \( K \) of \( M \) such that \( K \leq N \) there exists an element \( a \in L \) such that \( K = aN \).

Note that, \( a \in L \) is a multiplication element if and only if \( a \) is a weak meet principal element in \( L \) and \( N \) is a multiplication element if and only if \( N \) is a weak meet principal element in \( M \). We say that \( M \) is a multiplication lattice \( L \)-module if \( 1_M \) is a multiplication element in \( M \).

1.3. Theorem. Let \( L \) be a \( PG \)-lattice and \( M \) be a \( PG \)-lattice \( L \)-module. Then \( M \) is a multiplication lattice \( L \)-module if and only if for every maximal element \( q \in L \),

(i): For every principal element \( Y \in M \), there exists a principal element \( qY \in L \) with \( qY \leq q \) such that \( qY = 0_M \) or

(ii): There exists a principal element \( X \in M \) and a principal element \( b \in L \) with \( b \leq q \) such that \( b1_M \leq X \).
Proof. [see 16, Theorem 4]. □

1.4. Theorem. Let $L$ be a PG-lattice, and $M$ be a faithful multiplication PG-lattice $L$-module. Then the following conditions are equivalent.

(i): $1_M$ is a compact element of $M$.

(ii): If $a, c \in L$ such that $a1_M \leq c1_M$, then $a \leq c$.

(iii): For each element $N$ of $M$ there exists a unique element $a$ of $L$ such that $N = a1_M$.

(iv): $1_M \neq a1_M$ for any proper element $a$ of $L$.

(v): $1_M \neq p1_M$ for any maximal element $p$ of $L$.

Proof. [see 16, Theorem 5]. □

1.5. Proposition. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module such that $1_M$ compact. If $a \in L$ is a multiplication element, then $a1_M \in M$ is a multiplication element.

Proof. Let $K \leq a1_M$. Since $M$ is a multiplication module, $K = b1_M$ for some $b \in L$. Then $K = b1_M \leq a1_M$. Since $1_M$ is compact, $b \leq a$ by Theorem 2(ii). Since $a \in L$ is a multiplication element, we have $b = ac$ for some $c \in L$ and so $K = b1_M = (ac)1_M = c(a1_M)$. Consequently, $a1_M$ is a multiplication element. □

1.6. Proposition. Let $L$ be a PG-lattice and $M$ a faithful multiplication PG-lattice $L$-module such that $1_M$ is compact.

(i): $N$ is a multiplication element in $M$ if and only if $(K : L) N (N : L 1_M) = (K : L 1_M)$ for all $K \leq N$.

(ii): $a = (a1_M : L 1_M)$ for all $a \in L$.

(iii): $N$ is a multiplication element in $M$ if and only if $(N : L 1_M)$ is a multiplication element in $L$.

(iv): $a1_M$ is a multiplication element in $M$ if and only if $a$ is a multiplication element in $L$.

Proof. (i) $\Rightarrow$: Let $N$ be a multiplication element in $M$ and $K \leq N$. Then $K = bN$ for some $b \in L$. Since $M$ is a multiplication lattice $L$-module,

\[ K = bN = (bN : L 1_M) N = (bN : L N) (N : L 1_M) 1_M = (K : L 1_M) 1_M. \]

Therefore $(K : L N) (N : L 1_M) 1_M = (K : L 1_M) 1_M$.

$\Leftarrow$: Since

\[ K = (K : L 1_M) 1_M = (K : L N) (N : L 1_M) 1_M = (K : L N) N \]

for all $K \leq N$, $N$ is a multiplication element.

(ii) Since $M$ is a multiplication lattice module, we have $a1_M = (a1_M : L 1_M) 1_M$ and so $a = (a1_M : L 1_M)$ for all $a \in L$ by Theorem 2(ii).

(iii) $\Rightarrow$: Let $N$ be a multiplication element. If $a \leq (N : L 1_M)$, then $a = (a1_M : L 1_M)$ by (ii) and $a = (a1_M : L 1_M) = (a1_M : L N) (N : L 1_M)$ by (i). Therefore, $a = c(N : L 1_M)$ where $c = (a1_M : L N)$. Then $(N : L 1_M)$ is a multiplication element in $L$.

$\Leftarrow$: Let $(N : L 1_M)$ be a multiplication element in $L$. Then $(N : L 1_M) 1_M = N$ multiplication element in $M$ by Proposition 1.

(iv) $\Rightarrow$: Let $N = a1_M$ be a multiplication element in $M$. Then $(N : L 1_M) = (a1_M : L 1_M) = a$ is a multiplication element in $L$ by (iii).

$\Leftarrow$: Let $a \in L$ be a multiplication element in $L$. Then $N = a1_M$ is a multiplication element in $M$ by Proposition 1. □
1.7. Proposition. Let $L$ be a multiplicative lattice and $M$ a multiplication lattice $L$-module. If $L$ is a Noetherian (Artinian) lattice, then $M$ is a Noetherian (Artinian) $L$-module.

Proof. Suppose that $N_1 \leq N_2 \leq \ldots$ and $L$ is Noetherian. Then $(N_1 :_L 1_M) \leq (N_2 :_L 1_M) \leq \ldots$. Since $L$ is Noetherian, there is a positive integer $k > 0$ such that $(N_k :_L 1_M) = (N_{k+1} :_L 1_M) = \ldots$ and so $(N_k :_L 1_M)1_M = (N_{k+1} :_L 1_M)1_M = \ldots$. Therefore, $N_k = N_{k+1} = \ldots$. Similarly, if $L$ is Artinian, then $M$ is Artinian. □

1.8. Definition. Let $L$ be a multiplicative lattice and $M$ be a lattice $L$-module. Let $K$ be a proper element in $M$. $K$ is said to be a small element if for every element $N$ of $M$ such that $K \lor N = 1_M$ implies $N = 1_M$.

1.9. Definition. Let $L$ be a multiplicative lattice and $M$ be a lattice $L$-module. If every proper element of $M$ is small, then $M$ is called a hollow $L$-module.

1.10. Theorem. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $1_M$ compact. Then $M$ is a hollow $L$-module if and only if $L$ is a hollow $L$-module.

Proof. $\Rightarrow$: Suppose that $L$ is hollow. Let $a < 1_L$ such that $a \lor b = 1_L$ for some $b \in L$. Then $(a \lor b)1_M = a1_M \lor b1_M = 1_M$. Since $a < 1_L$, $a1_M < 1_M$ by Theorem 2. By hypothesis, $b1_M = 1_M$ and hence $b = 1_L$ by Theorem 2 (ii). Therefore $a$ is a small element in $L$.

$\Leftarrow$: Suppose that $L$ is a hollow $L$-module. Let $N < 1_M$ and $K$ be any element in $M$ such that $N \lor K = 1_M$. Since $M$ is a multiplication $L$-module, we have $N = (N :_L 1_M)1_M$ and $K = (K :_L 1_M)1_M$. Then,

$$1_M = N \lor K = (N :_L 1_M)1_M \lor (K :_L 1_M)1_M = ([N :_L 1_M] \lor (K :_L 1_M))1_M.$$ 

Therefore, $(N :_L 1_M) \lor (K :_L 1_M) = 1_L$ by Theorem 2 (ii). Since $N = (N :_L 1_M)1_M < 1_M$, we have $(N :_L 1_M) < 1_L$ and so $(K :_L 1_M) = 1_L$ by hypothesis. This shows that $K = 1_M$. Consequently, $M$ is hollow. □

1.11. Theorem. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $1_M$ compact. Then, $N$ is small if and only if there exists a small element $a \in L$ such that $N = a1_M$.

Proof. $\Rightarrow$: Suppose that $N \in M$ is small and $N = a1_M$ for some proper element $a$ in $L$. Suppose that $a \lor b = 1_L$ for some $b \in L$. Then

$$N \lor b1_M = a1_M \lor b1_M = (a \lor b)1_M = 1_M$$

and so $b1_M = 1_M$ by hypothesis. Hence $b = 1_L$ by Theorem 2. This shows that $a$ is small in $L$.

$\Leftarrow$: Suppose that $a \in L$ is small such that $N = a1_M$. Let $N \lor K = a1_M \lor K = 1_M$ for some $K \in M$. Since $M$ is a multiplication $L$-module, there is an element $b \in L$ such that $K = b1_M$ and hence $(a \lor b)1_M = a1_M \lor K = 1_M$. Then $a \lor b = 1_L$ by Theorem 2 (ii) and hence $b = 1_L$ by hypothesis. Therefore, $K = 1_M$. This shows that $a1_M$ is small. □

1.12. Definition. Let $M$ be a $L$-module. An element $N < 1_M$ in $M$ is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ i.e. $a \leq (N :_L 1_M)$ for every $a \in L, X \in M$.

1.13. Definition. Let $M$ be an $L$-module. $M$ is said to be prime $L$-module if $0_M$ is prime element of $M$.

It is clear that $0_M$ is prime element in $M$ if and only if $(0_M :_L 1_M) = (0_M :_L N)$ for all $0_M \neq N \in M$. 

1.14. Definition. Let $M$ be an $L$-module. $M$ is said to be coprime $L$-module if $(0_M :_L 1_M) = (N :_L 1_M)$ for all $N \in M$ such that $N < 1_M$.

Recall that a lattice $L$-module $M$ is called simple if $M = \{0_M, 1_M\}$.

1.15. Proposition. If $M$ is a multiplication and coprime $L$-module, then $M$ is simple.

Proof. Let $N \in M$ such that $N < 1_M$. Since $M$ is a coprime $L$-module, we have $(0_M :_L 1_M) = (N :_L 1_M)$. Since $M$ is a multiplication $L$-module, it follows that $N = (N :_L 1_M) 1_M = (0_M :_L 1_M) 1_M = 0_M$. Then $M$ is simple. □

1.16. Definition. Let $L$ be a $PG$-lattice and $M$ a $PG$-lattice $L$-module. Let $p$ be a maximal element of $L$. $M$ is called $p$-torsion provided for each principal element $X \in M$ there exists a principal element $qX \in L$, $qX \not\subseteq p$ such that $qX = 0_M$.

1.17. Definition. Let $L$ be a $PG$-lattice and $M$ a $PG$-lattice $L$-module. $M$ is called $p$-cyclic provided there exists a principal element $Z \in M$ and a principal element $q \in L$, $q \not\subseteq p$ such that $q1_M \leq Z$.

Let $M$ be an $L$-module. Let $N$ and $K$ be elements of $M$ such that $N \leq K$. Define $[N, K] = \{A \in M : N \leq A \leq K\}$. Then $[N, K]$ is an $L$-module. It is clear that $N$ is a multiplication element if and only if $[0_M, N]$ is a multiplication lattice $L$-module. Recall that $\text{ann}(X) = (0_M :_L X)$ for any $X \in M$.

1.18. Theorem. Let $L$ be a $PG$-lattice and $M$ a $PG$-lattice $L$-module. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of $M$ such that $N = \bigvee_{\lambda \in \Lambda} N_\lambda$ and $a = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L N)$.

i) $N$ is a multiplication element in $M$.

ii) $H = aH$ for all elements $H \leq N$.

iii) $1_L = a\bigvee \text{ann}(X)$ for all principal elements $X \leq N$.

iv) $(N :_L K) \bigvee \text{ann}(X) = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L K) \bigvee \text{ann}(X)$ for all principal elements $X \leq N$ and for all elements $K$ in $M$.

Then, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). If all $N_\lambda$ are multiplication elements, then the conditions are equivalent.

Proof. (i) $\Rightarrow$ (ii). Let $a = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L N)$. Then $aN = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L N) = N$. Since $N$ is a multiplication element, there exist an element $h \in L$ such that $H = hN$ for $H \leq N$.

Therefore, $aH = ahN = h(aN) = hN = H$.

(ii) $\Rightarrow$ (iii). Suppose that $1_L \neq a\bigvee \text{ann}(X)$ for all principal elements $X \leq N$. There exists a maximal element $p \in L$ such that $a\bigvee \text{ann}(X) \leq p$ for each principal element $X \leq N$. Since $X = aX \leq pX \leq X$, we have $X = pX$. Then, $1_L = (pX :_L X) = p\bigvee \text{ann}(X)$.

Since $\text{ann}(X) \leq a\bigvee \text{ann}(X) \leq p$, we get a contradiction.

(iii) $\Rightarrow$ (ii). Since $X = (a\bigvee \text{ann}(X)) X = aX$ for all principal elements $X \leq N$, it follows that $H = aH$ for every $H \leq N$.

(iii) $\Rightarrow$ (iv). Since $\bigvee_{\lambda \in \Lambda} (N_\lambda :_L K) K \leq \bigvee_{\lambda \in \Lambda} N_\lambda = N$, we have $\bigvee_{\lambda \in \Lambda} (N_\lambda :_L K) \leq \bigvee_{\lambda \in \Lambda} N_\lambda :_L K = (N :_L K)$. Therefore, $\bigvee_{\lambda \in \Lambda} (N_\lambda :_L K) \bigvee \text{ann}(X) \leq (N :_L K) \bigvee \text{ann}(X)$ for all principal elements $X \leq N$. Conversely, $w_1$ and $w_2$ be principal elements such that $w_1 \bigvee w_2 \leq (N :_L K) \bigvee \text{ann}(X)$ where $w_1 \leq (N :_L K)$ and $w_2 \leq \text{ann}(X)$. Then $1_L = a\bigvee \text{ann}(X) = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L N) \bigvee \text{ann}(X)$. Hence $w_1 = \bigvee_{\lambda \in \Lambda} (N_\lambda :_L N) w_1 \bigvee \text{ann}(X) w_1$.

Since $w_1 K \leq N$, we have $(N_\lambda :_L N) w_1 K \leq (N_\lambda :_L N) N \leq N$ and so $(N_\lambda :_L N) w_1 \leq (N_\lambda :_L K)$. Therefore,
\[
\begin{align*}
 w_1 \lor w_2 &= \bigvee_{\lambda \in \Lambda} (N_\lambda : L N) w_1 \lor \text{ann}(X) w_2 \\
 &\leq \bigvee_{\lambda \in \Lambda} (N_\lambda : L K) \lor \text{ann}(X).
\end{align*}
\]

Hence, \((N : L K) \lor \text{ann}(X) \leq \bigvee_{\lambda \in \Lambda} (N_\lambda : L K) \lor \text{ann}(X)\).

(iv) \implies (iii). If we take \(K = N\), then \(1_L = (N : L N) \lor \text{ann}(X) = a \lor \text{ann}(X)\)

(iii) \implies (i). Suppose that \(1_L = a \lor \text{ann}(X)\) for all principal elements \(X \leq N\). Suppose that \(N\) is not \(p\)-torsion. There exists a principal element \(X \leq N\) such that \(\text{ann}(X) = (0_M : L X) \leq p\). Since \(1_L = a \lor \text{ann}(X)\), we have \(a = \bigvee_{\lambda \in \Lambda} (N_\lambda : L N) \neq p\). There exists \(\lambda \in \Lambda\) such that \((N_\lambda : L N) \neq p\). There exists a principal element \(b \neq p\) such that \(b \leq (N_\lambda : L N)\). Since \(N_\lambda\) is a multiplicative element and not \(p\)-torsion, it follows that \(N_\lambda\) is \(p\)-cyclic by Theorem 1. Indeed, if \(N_\lambda\) is a \(p\)-torsion, then \(cN_\lambda = 0_M\) for some principal element \(c \neq p\) and so \(bN \leq N_\lambda\) implies \(bcN \leq cN_\lambda = 0_M\). Then \(bc \leq (0_M : L N)\).

Therefore \(bcY = 0_M\) for all principal elements \(Y \leq N\) and principal element \(bc \neq p\). Since \(N\) is not \(p\)-torsion, this is a contradiction. Hence \(N_\lambda\) is \(p\)-cyclic. Therefore, \(dN_\lambda \leq Y_\lambda\) for some principal element \(Y_\lambda \leq N_\lambda\) and principal element \(d \neq p\).

Therefore, \(bN \leq N_\lambda\) and so \(bdN \leq dN_\lambda \leq Y_\lambda\) and \(bd \neq p\). Consequently, \(N\) is not \(p\)-cyclic. □

1.19. Theorem. (Nakagama Lemma) Let \(M\) be a non-zero multiplication \(PG\)-lattice \(L\)-module. Let \(a \in L\) such that for all maximal element \(q \in L\), \(a \leq q\). Then \(a1_M < 1_M\).

Proof. Let \(a \in L\) such that \(a \leq q\) for all maximal element \(q \in L\) and suppose that \(a1_M = 1_M\). Let consider principal element \(0_M \neq X \in M\). Since \(M\) is a multiplication \(L\)-module, we have \(X = b1_M\) for some \(b \in L\). Hence \(X = b1_M = ab1_M = aX\). Thus \(1_L = (aX : X) = a \lor (0_M : L X)\) for all principal elements \(X \in M\). Since \(0_M : L X < 1_L\), there exists a maximal element \(p \in L\) such that \((0_M : L X) \leq p\). By hypothesis \(a \leq p\), hence \(1_L = (aX : X) = a \lor (0_M : L X) \leq p\) and we obtain a contradiction. □

1.20. Proposition. Let \(L\) be a multiplicative \(PG\)-lattice. Let \(M\) be a multiplication \(PG\)-lattice \(L\)-module and \((0_M : 1_M) \leq p\) for some prime element \(p \in L\). If \(a1_M \leq p1_M\) for some \(a \in L\), then \(a \leq p\) or \(p1_M = 1_M\).

Proof. If \(a1_M \leq p1_M\) for some \(a \in L\), then \(aX \leq p1_M\) for all principal element \(X \in M\). Hence \(a \leq p\) or \(X \leq p1_M\) [see 16, Theorem 6]. If \(a \neq p\), then \(X \leq p1_M\) for all principal element \(X \in M\), hence it is clear that \(p1_M = 1_M\). □

1.21. Proposition. Let \(L\) be a multiplicative \(PG\)-lattice. Let \(M\) be a multiplication \(PG\)-lattice \(L\)-module. Then \(K\) is maximal element of \(M\) if and only if there is a maximal \(p \in L\) such that \(K = p1_M < 1_M\).

Proof. \(\Leftarrow\): If there exist a maximal element \(p \in L\) such that \(K = p1_M < 1_M\), then \(K\) is maximal [see 16, Proposition 4].

\(\Rightarrow\): Let \(K\) be a maximal element of \(M\) and \((K : 1_M) = q\). Since \(M\) is a multiplication lattice module, we have \(K = q1_M\). We show that \((K : 1_M) = q\) is a maximal element of \(L\). If \(q\) is not a maximal element, there exists a maximal element \(p\) such that \(q < p\). Then \(p1_M \neq K\). Indeed, if \(p1_M \leq K\), then \(p \leq (K : 1_M) = q\). This is a contradiction. Therefore, \(1_M = K \lor p1_M = (q \lor p)1_M = p1_M\). Hence \(X = pX\) for all principal elements \(X\) and so \(1_L = (pX : L X) = p \lor (0_M : L X)\). This implies that \((0_M : L X) \neq p\).

Therefore, there exists a principal element \(pX \in L\) such that \(pX \leq (0_M : L X)\) and \(pX \neq p\). If we take a principal element \(X\) such that \(X \neq K\), then \(X \lor K = 1_M\). Hence \(pX \lor pXK = pX1_M\) and so \(pXK = pX1_M \leq K\). Therefore, \(pX \leq (K : 1_M) = q < p\). This is a contradiction. □
1.22. Theorem. Let $L$ be a CG-lattice and $M$ be a PG-lattice $L$-module. Then $M$ is a multiplication lattice $L$-module if and only if for every maximal element $q \in L$,

(i) For every principal element $Y \in M$, there exists a compact element $q_Y \in L$ with $q_Y \not\leq q$ such that $q_Y Y = 0_M$ or

(ii) There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \not\leq q$ such that $b_1M \leq X$.

Proof. $\implies$: Let $M$ be a multiplication lattice $L$-module. We have two cases.

Case 1. Let $q_1M = 1_M$ where $q$ is a maximal element of $L$. For every principal element $Y \in M$, there exists an element $a \in L$ such that $Y = a_1M$. Then $Y = a_1M = aq_1M = qY$. Therefore, $aL = (qY : L)Y = q \lor (0M : L)Y$. Hence $(0M : L)Y \not\leq q$. There exists a compact element $q_Y$ such that $q_Y \not\leq (0M : L)Y$ and $q_Y \not\leq q$.

Case 2. Let $q_1M < 1_M$. There exists a principal element $X \in M$ such that $X = j_1M \not\leq q_1M$, with $j \in L, j \not\leq q$. There exists a compact element $b \in L$ with $b \leq j$ and $b \not\leq q$. We obtain $b_1M \leq j_1M = X$.

$\impliedby$: Let $N \in M$. Put $a = (N : L)M$. Clearly $a_1M = (N : L)1_MM \leq N$. Take any principal element $Y \leq N$. We will show that $(a_1M : L)Y = 1_L$. Suppose there exists a maximal element $q \in L$ such that $(a_1M : L)Y \leq q$. We have two cases.

Case 1. Suppose that $(i)$ is satisfied. There exists a compact element $q_Y \in L$ with $q_Y \not\leq q$ such that $q_Y Y = 0_M$ for every principal element $Y \in M$. Then $q_Y \not\leq (0M : L)$ $Y \iff (a_1M : L)Y \leq q$. This is a contradiction.

Case 2. Suppose that $(ii)$ is satisfied. There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \not\leq q$ such that $b_1M \leq X$. Then $bN \leq b_1M \leq X$ for any $N \in M$. Since $X$ is a principal element of $M$, $bN = (bN : L)X$. Then $b(bN : L)X \leq (a_1M : L)X = bN \leq N$ and so $b(bN : L)X \leq a = (N : L)1_M$. Therefore $b_2Y \leq b_2bN \leq b_2bN : L)X \leq aX \leq a_1M$. Since $b_2 \leq (a_1M : L)Y \leq q$. Since $q$ is maximal (and so prime) element of $L$, we have $b \leq q$. This is a contradiction.

\hfill $\Box$

1.23. Definition. Let $M$ be an $L$-module. An element $N < 1_M$ in $M$ is said to be primary, if $aX \leq N$ and $X \not\leq N$ implies $a^k1_M \leq N$, for some $k \geq 0$ i.e. $a^k \leq (N : 1_M)$ for every $a \in L, X \in M$.

If $a$ is an element of a multiplicative lattice $L$, we define $\sqrt{a} = \lor \{b \in L : b^k \leq a \text{ for some natural number } n\}$.

1.24. Theorem. Let $L$ be a CG-lattice and $M$ be a multiplication PG-lattice $L$-module. Suppose that $p$ is a primary element in $L$ with $(0M : L)1_M \leq p$. If $aX \leq pL$, where $a \in L, X \in M$, then $X \leq p1_M$ or $a \leq \sqrt{p}$.

Proof. We may suppose that $X$ is principal in $M$. Suppose that $aX \leq p1_M$ with $a \not\leq \sqrt{p}$. We will show that $(p1_M : L)X = 1_L$. Suppose that there exists a maximal element $q \in L$ such that $(p1_M : L)X \leq q$. By theorem 7, we have two cases.

Case 1. If there exists a compact element $q_a \in L$ with $q_a \not\leq q$ such that $0_M = q_aX$, then $q_a \leq (0_M : L)X \leq (p1_M : L)X \leq q$. This is a contradiction.

Case 2. If there exists a principal element $Y \in M$ and a compact element $b \in L$ with $b \not\leq q$ such that $b_1M \leq Y$, then $bX \leq b_1M \leq Y$. Since $Y$ is principal, we have $bX = (bX : L)Y$. Put $(bX : L)Y = s$. Then $abX = asY$. Since $Y$ is join-principal, it follows that $(asY : L)Y = as \lor (0_M : L)Y$. Since $Y$ is meet-principal, we have $abX = (abX : L)Y$. Put $c = (abX : L)Y$. Since $cY = abX \leq bp1_M \leq pY$, it follows that $c \lor (0_M : L)Y = (cY : L)Y \leq (pY : L)Y = p \lor (0_M : L)Y$. Since $b(0_M : L)1_M = (0_M : L)1_M = (0_M : L)Yb_1M \leq (0_M : L)Y = 0_M$, we have $b(0_M : L)Y \leq (0_M : L)1_M \leq p$. Hence $bc \lor b(0_M : L)Y \leq bp \lor b(0_M : L)Y \leq p$. Therefore, $bc \leq p$. On the other hand, $c = (abX : L)Y = (asY : L)Y = as \lor (0_M : L)Y$ and so $abX \leq abY \lor b(0_M : L)Y = bc \leq p$.

If $b \leq \sqrt{p}$, since $b$ is compact, we obtain $b \leq a_1 \lor a_2 \lor \ldots \lor a_m$ such that $a_i \leq p$.
for each \( i = 1, \ldots, m \). For \( k = n_1 + n_2 + \ldots + n_m \), we have \( b^k \leq a_1^k \vee a_2^k \vee \ldots \vee a_m^k \leq p \). Then \( b^k \leq p \leq (p_1M :L X) \leq q \). Since \( q \) is maximal, we obtain \( b \leq q \). This is a contradiction. Therefore, \( b \not\subseteq \sqrt{p} \). Since \( a \not\subseteq \sqrt{p} \) and \( p \) is primary, we have \( a \leq p \). So, \( bX = sY \leq pY \leq p1M \) and therefore \( b \leq (p1M :L X) \leq q \). This is a contradiction.

1.25. Corollary. Let \( L \) be a CG-lattice and \( M \) be a PG-lattice \( L \)-module. Let \( M \) be a multiplication lattice \( L \)-module and \( N < 1_M \). Then the following condition are equivalent.

i) \( N \) is a primary element in \( M \).

ii) \( (N :L 1_M) \) is a primary element in \( L \).

iii) There exists a primary element \( p \) in \( L \) with \( (0_M :L 1_M) \leq p \) such that \( N = p1_M \).

Proof. i) \( \implies \) ii) \( \implies \) iii) : Clear.

iii) \( \implies \) i) : Let \( aX \leq N \) and \( X \not\subseteq N \) for \( a \in L, X \in M \). Since there exists a primary element \( p \) in \( L \) with \( (0_M :L 1_M) \leq p \) such that \( N = p1_M \), we have \( aX \leq p1_M \) and \( X \not\subseteq p1_M \). By theorem 8, \( a \leq \sqrt{p} \) and so \( a^k \leq p \) for some \( k > 0 \). Hence \( a^k \leq (p1M :L 1_M) = (N :L 1_M) \). □

REFERENCES


