Symmetry classification of variable coefficient cubic-quintic nonlinear Schrödinger equations

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A Lie-algebraic classification of the variable coefficient cubic-quintic nonlinear Schrödinger equations involving 5 arbitrary functions of space and time is performed under the action of equivalence transformations. It is shown that the symmetry group can be at most four-dimensional in the case of genuine cubic-quintic nonlinearity. It may be five-dimensional (isomorphic to the Galilei similitude algebra gs(1)) when the equation is of cubic type, and six-dimensional (isomorphic to the Schrödinger algebra sch(1)) when it is of quintic type. © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4789543]

I. INTRODUCTION

In this paper we are interested in giving a classification of variable coefficient cubic-quintic nonlinear Schrödinger (CQNLS) equations

\[ iu_t + f(x, t)u_{xx} + k(x, t)u_x + g(x, t)|u|^2u + q(x, t)|u|^4u + h(x, t)u = 0 \]  \hspace{1cm} (1.1)

according to their Lie point symmetries up to equivalence. Here \( u \) is a complex-valued function, \( f \) is real-valued, \( k, g, q, h \) are complex-valued functions of the form \( k = k_1(x, t) + ik_2(x, t), g(x, t) = g_1(x, t) + ig_2(x, t), q(x, t) = q_1(x, t) + iq_2(x, t) \), and \( h(x, t) = h_1(x, t) + ih_2(x, t) \). We assume that \( g \neq 0 \) or \( q \neq 0 \), that is, at least one of \( g_1, g_2, q_1, q_2 \) is different from zero. Equation (1.1) contains two physically important equations: the cubic Schrödinger equation for \( k = q = 0 \) and the quintic Schrödinger equation for \( k = q = 0 \) in one space dimension.

The motivation for this work is due to a wide variety of physical applications of equations fitting within the class (1.1). The role that the cubic nonlinear Schrödinger equation or simply nonlinear Schrödinger (NLS) equation and its variable coefficient extensions play in many areas of physics is well-known. In fact, there is a vast amount of literature devoted to the remarkable features (symmetries, integrability, analytical solutions, and qualitative behaviors) of these equations both in mathematical and physical contexts. One of the major applications of the NLS equation arises in nonlinear optics (for example, in the field of fiber-optic communication) in which only cubic nonlinearity is taken into account. On the other hand, when higher optical intensities or materials with higher order coefficients (e.g. semiconductor doped glasses) are considered, higher order nonlinearity becomes essential. The CQNLS equation has been around for many years as a remedy to cover problems in such situations. Recently, variable coefficient extensions of the CQNLS equation have been proposed as a more realistic model in nonlinear optics and other areas of physics. There exist a number of works devoted to both theoretical and experimental applications of the equations under study. Reference 1 considers the properties of bright and dark solitary wave solutions to the constant coefficient cubic-quintic equation for the normal dispersion region of

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strongly nonlinear optical fibers. In Ref. 2 we see that the CQNLS equation models the dynamics of optical solitons inside a nonlinear organic film. Reference 3 notes a pure quintic equation appearing in the theory of Bose-Einstein condensation, whereas Ref. 4 can also be mentioned in the same context. Reference 5 studies a cubic-quintic case where all the coefficients are allowed to be \( t \) dependent. In Ref. 6 the cubic equation has a nonlinearity coefficient depending on the variable \( x \). We note that in the context of optics variables \( t \) and \( x \) denote the propagation distance and the retarded time, respectively.

Symmetry classification of \((1.1)\) in the special case \( k = q = 0 \) was given in Ref. 7. Similar works based on a different approach appeared in Refs. 8 and 9. An in-depth analysis of the constant coefficient version of \((1.1)\) with \( k = 0 \) in \( 3 + 1 \)-dimensions was carried out in a series of papers by the authors studied Lie point symmetries and gave a complete subalgebra classification of symmetry algebras, reductions, and a comprehensive analysis of the explicit (group-invariant) Lie point symmetries. Other symmetry classification results relevant to several one- and multi-dimensional versions of the nonlinear Schrödinger equations involving arbitrary functions depending not only on space-time variables but also on dependent variables and its space derivatives can be found, for example, in Refs. 10–16.

The works mentioned above do not include the \( ku_{x} \) derivative term of \((1.1)\). For this case we can mention Refs. 17 and 18 which analyze the integrability properties and solutions of the cubic equation with less general coefficients than those treated here. Specifically, the case \( f = 1, k = (n - 1)/x \) corresponds to the radial counterpart of the cubic and quintic equations in \( n \) space dimensions with \( x \) playing the role of the radial coordinate when \( n \geq 2 \) and real half-line when \( n = 1 \). We should add that a multi-dimensional extension is not relevant in the optical context.

II. THE EQUIVALENCE GROUP

By definition, the equivalence group of \((1.1)\) is the group of transformations preserving the form of \((1.1)\). This is given in the following definition.

Definition 1: The equivalence group \( E \) of Eq. \((1.1)\) is the group of smooth transformations \((t, x, u) \rightarrow (\tilde{t}, \tilde{x}, \tilde{u})\) preserving the differential structure. More precisely, \( E \) maps \((1.1)\) to

\[
i\tilde{u}_{t} + \tilde{f} \tilde{u}_{x} + \tilde{k} \tilde{u}_{x} + \tilde{g} |\tilde{u}|^{2} \tilde{u} + \tilde{q} |\tilde{u}|^{4} \tilde{u} + \tilde{h} \tilde{u} = 0.
\]

(2.1)

The equivalence group leaves the differential terms invariant but changes the coefficients, namely it leaves the equation form-invariant. A point symmetry group is a subgroup of \( E \) obtained when the coefficients remain unchanged under \( E \). These transformations are sometimes called allowed or admissible transformations.

Two approaches can be taken to find \( E \). One is the infinitesimal method and requires solving a large system of overdetermined linear partial differential equations just like determining the infinitesimal symmetries. The disadvantage of this approach is that the discrete equivalence group does not come up as a subgroup. The other is direct approach and will be used below.

Proposition 1: The equivalence group \( E \) of Eq. \((1.1)\) is given by

\[
E : \quad \tilde{t} = T(t), \quad \tilde{x} = X(x, t), \quad u = Q(x, t) \tilde{u}(\tilde{x}, \tilde{t}),
\]

(2.2)

where the coefficients transform by

\[
\tilde{f} = \frac{fX^{2}}{T}, \quad \tilde{g} = \frac{g|Q|^{2}}{T}, \quad \tilde{q} = \frac{q|Q|^{4}}{T}.
\]

(2.3a, 2.3b, 2.3c)
\[ \tilde{h} = \frac{1}{T} \left( h + i \frac{Q_t}{Q} + f \frac{Q_{x_t}}{Q} + k \frac{Q_x}{Q} \right), \]
\[ \tilde{k} = \frac{1}{T} \left( iX_t + fX_{x_t} + 2fX_x Q_x \right) \]
under the condition \( \tilde{X}, \tilde{T}, Q \neq 0 \).

For the sake of convenience we introduce the following moduli and phases for the complex functions \( Q, u, \) and \( \tilde{u} \)
\[ Q(x, t) = R(x, t) e^{iR(x, t)}, \quad u = \rho(x, t) e^{i\omega(x, t)}, \quad \tilde{u} = \tilde{\rho}(\tilde{x}, \tilde{t}) e^{i\tilde{\omega}(\tilde{x}, \tilde{t})}. \]

### A. Canonical cubic-quintic nonlinear Schrödinger equation

We can use the equivalence group to transform (1.1) to some canonical form by choosing the free functions in the transformation suitably. Indeed, first of all, one can normalize \( f \rightarrow 1 \) by restricting \( X \) to
\[ X(x, t) = \epsilon \sqrt{T} x + \xi(t), \quad \epsilon = \mp 1. \]

With this choice of \( X, \tilde{k} \) can be made zero by taking \( R \) and \( \theta \) as solutions of the following equations
\[ 2\frac{R_x}{R} + k_1 = 0, \quad 2X_x \theta_x + X_t + k_2X_x = 0, \]
so that the canonical form of (1.1) is
\[ iu_t + u_{xx} + g(x, t)|u|^2u + q(x, t)|u|^4u + h(x, t)u = 0. \]

**Corollary 1:** The equivalence group of the canonical equation (2.6) with \( T, R_0, \xi, \) and \( \eta \) being arbitrary functions of \( t \)
\[ E : \quad \tilde{i} = T(t), \quad \tilde{x} = \epsilon \sqrt{T} x + \xi(t), \quad \epsilon = \mp 1, \quad u = R_0(t) e^{i\theta(x, t)} \tilde{u}, \]
where
\[ \dot{T} \neq 0, \quad R_0(t) \neq 0, \quad \theta(x, t) = \frac{T}{8T} x^2 - \frac{\xi}{2\epsilon \sqrt{T}} x + \eta(t) \]
and the transformed new coefficients are
\[ \tilde{g} = \frac{g R_0^2}{T}, \quad \tilde{q} = \frac{q R_0^4}{T}, \quad \tilde{h} = \frac{1}{T} \left[ (h_1 - \theta_1 - \theta_x^2) + i \left( h_2 \frac{R_0}{R_0} + \theta_x \right) \right]. \]

Notice that since \( \theta \) is a second degree polynomial in \( x \) and \( R_0 \) depends on only \( t \), not every potential function \( h \) can be killed through these transformations. We recall the following relations
\[ \tilde{\rho}(\tilde{x}, \tilde{t}) = \frac{\rho(x, t)}{R_0(t)}, \quad \tilde{\omega}(\tilde{x}, \tilde{t}) = \omega(x, t) - \theta(x, t). \]

**Remark 1:** If \( g \) and \( q \) have no dependence on \( x \) we can always set \( g_1 \rightarrow \pm 1 \) and \( q_1 \rightarrow \pm 1 \).

### III. SYMMETRY ALGEBRA

The symmetry algebra is generated by vector fields of the form
\[ Q = \xi(x, t, \rho, \omega) \partial_x + \tau(x, t, \rho, \omega) \partial_t + \Phi(x, t, \rho, \omega) \partial_\rho + \Psi(x, t, \rho, \omega) \partial_\omega. \]

The standard Lie infinitesimal algorithm requires calculating the second order prolongation of \( Q \) to the jet space \( J^2(\mathbb{R}^2, \mathbb{R}^2) \) with local coordinates \( (t, x, \rho, \omega) \) and derivatives up to and including second
order. The vector field $Q$ becomes an infinitesimal symmetry of the equation when its 2nd order prolongation annihilates Eq. (2.6) written as a system in terms of $\rho$, $\omega$ on its solution manifold. The symmetry criterion provides a set of overdetermined linear partial differential equations. Solving those equations not involving coefficients we obtain the following assertion.

**Proposition 2:** The symmetry algebra of the canonical CQNLS equation is generated by the vector fields of the form

$$Q = \chi(x,t)\partial_x + \tau(t)\partial_t + A(t)\rho\partial_\rho + D(x,t)\partial_\omega,$$

where the functions $\chi$ and $D$ are defined by

$$\chi(x,t) = \frac{\tau}{2}x + \alpha(t), \quad D(x,t) = \frac{\tau}{8}x^2 + \frac{\alpha}{2}x + n(t)$$

and the coefficients in the equation satisfy the determining equations

$$\begin{align*}
\tau g_{1,t} + \chi g_{1,x} + (2A + \tau)g_1 &= 0, \\
\tau g_{2,t} + \chi g_{2,x} + (2A + \tau)g_2 &= 0, \\
\tau q_{1,t} + \chi q_{1,x} + (4A + \tau)q_1 &= 0, \\
\tau q_{2,t} + \chi q_{2,x} + (4A + \tau)q_2 &= 0, \\
\tau h_{1,t} + \chi h_{1,x} + \tau h_1 - D_t &= 0, \\
\tau h_{2,t} + \chi h_{2,x} + \tau h_2 + A + \frac{\tau}{4} &= 0.
\end{align*}$$

**A. Symmetries for the equations with constant coefficients**

Equations (3.3) are straightforward to solve in the special case where all the coefficients are constants

$$g(x,t) = g_1 + ig_2, \quad q(x,t) = q_1 + iq_2, \quad h(x,t) = h_1 + ih_2.$$  

We sum up the results as follows. The case $h_2 = 0$:

1. $g \neq 0, q \neq 0$ (genuine cubic-quintic case): The symmetry algebra is four-dimensional and is spanned by

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_x + \frac{x}{2}\partial_\omega, \quad Q_4 = \partial_\omega.$$

It is solvable and isomorphic to the one-dimensional Galilei algebra $\mathfrak{gal}(1)$.

2. $q = 0, g \neq 0$ (cubic case): The symmetry algebra is five-dimensional and has the basis

$$Q_1 = \partial_t + h_1\partial_\omega, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_x + \frac{x}{2}\partial_\omega, \quad Q_4 = \partial_\omega,$$

$$Q_5 = \frac{x}{2}\partial_x + t\partial_t - \frac{1}{2}\rho\partial_\rho + h_1t\partial_\omega.$$

It is solvable and isomorphic to the one-dimensional Galilei similitude algebra $\mathfrak{gs}(1) \simeq \mathfrak{h}(1) \cup \langle Q_1, Q_5 \rangle$ with $\mathfrak{h}(1) = \langle Q_2, Q_3, Q_4 \rangle$ being the nilpotent ideal (Heisenberg algebra) and $\cup$ denoting the semi-direct sum of algebras.
B. One- and two-dimensional symmetry algebras

\[ q \neq 0, \ g = 0 \] (quintic case): The symmetry algebra is six-dimensional and is spanned by
\[ Q_1 = \partial_t + h_1 \partial_\omega, \quad Q_2 = \partial_x, \quad Q_3 = t \partial_x + \frac{x}{2} \partial_\omega, \quad Q_4 = \partial_\omega, \quad Q_5 = \frac{x}{2} \partial_x + t \partial_x - \frac{1}{4} \rho \partial_\rho + h_1 t \partial_\omega, \quad Q_6 = x t \partial_x + t^2 \partial_t - \frac{1}{2} t \rho \partial_\rho + \left(\frac{x^2}{4} + h_1 t^2\right) \partial_\omega. \] (3.6)

It is non-solvable and isomorphic to the one-dimensional Schrödinger algebra \( \mathfrak{sch}(1) \) having the Levi decomposition
\[ \mathfrak{sch}(1) \cong (Q_1, Q_5, Q_6) \cong \mathfrak{h}(1). \]
The simple algebra \( (Q_1, Q_5, Q_6) \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \).

The symmetry algebra allowed in the case \( h = h_1 + ih_2 \) with \( h_2 \neq 0 \) is at most four-dimensional and isomorphic to \( (3.4) \) for any \( g \) and \( q \).

B. One- and two-dimensional symmetry algebras

**Proposition 3:** The vector field \((3.1)\) can be transformed by the transformations \((2.7)\) to one of the following canonical forms, in other words, there are precisely three inequivalent realizations:
\[ Q = \partial_\omega, \quad Q = \partial_t, \quad Q = \partial_x + m(t) \rho \partial_\rho. \] (3.7)

The phase-shift symmetry group \( \omega \rightarrow \omega + \epsilon \) generated by \( Q_1 = \partial_\omega \) leaves the equation invariant for any choice of the coefficients.

**Proof:** There are three different cases:

(i) \( \tau(t) = \alpha(t) = 0 \): We have \( Q = A(t) \rho \partial_\rho + n(t) \partial_\omega \). Since at least one of \( g_1, g_2, q_1, q_2 \) is different from zero, one of the four equations \((3.3a)-(3.3d)\) imposes \( A(t) = 0 \), which means that the other three are satisfied identically. Equation \((3.3e)\) is satisfied if \( n(t) \) is a constant and \((3.3f)\) automatically holds. Therefore \( Q \) simplifies to
\[ Q = \partial_\omega \] (3.8)
as a canonical form of a one-dimensional algebra. It is usually called the “gauge symmetry” as it consists of pure gauge transformations shifting only the phase while leaving all other coordinates invariant. In this case there are no restrictions on the coefficients \( g(x, t), q(x, t) \), and \( h(x, t) \).

(ii) \( \tau(t) \neq 0 \): We can transform \( Q \) into \( \partial_t \) by choosing
\[ \tilde{T} = \tau^{-1}, \quad \tilde{\xi} = -\frac{\epsilon \alpha}{\tau^{3/2}}, \quad R_0(t) = r_0 \exp \left( \int \frac{A}{\tau} dt \right), \quad \tilde{\eta} = \frac{n}{\tau} - \frac{\alpha^2}{2\tau^2}, \]
where \( r_0 \) is a constant, in the allowed transformation \((2.7)\). The canonical form of \( Q \) is
\[ Q = \partial_t. \] (3.9)
The corresponding equation contains coefficients depending only on \( x \). \( Q \) remains invariant under the transformation
\[ \mathcal{E} : \quad \tilde{x} = \epsilon x + \xi_0, \quad \tilde{t} = t + T_0, \quad \tilde{\rho} = \frac{\rho}{r_0}, \quad \tilde{\omega} = \omega - \eta_0, \] (3.10)
where \( \xi_0, T_0, \eta_0 \) are constants.

(iii) \( \tau(t) = 0, \alpha(t) \neq 0 \): Similar arguments can be used to find the canonical vector field. The coefficients figuring in the invariant equation are
\[ g(x, t) = (g_1(t) + ig_2(t))e^{-2\lambda m(t)}, \] (3.11a)
C. Three-dimensional algebras

1. Abelian algebras

A real three-dimensional algebra is either simple or solvable. The only three-dimensional simple algebras are sl(2, \mathbb{R}) and so(3, \mathbb{R}). The first algebra contains a two-dimensional non-Abelian algebra. This implies that there can be no sl(2, \mathbb{R}) realizations. We can also show that so(3, \mathbb{R}) cannot be realized in terms of vector fields (3.1). So any Lie symmetry algebra L with dimension \( \text{dim } L \geq 2 \) contains \( Q_1 = \partial_x \) as a subalgebra.

1. Abelian algebras

We let \( Q_1 = \partial_x, Q_2 = Q \) of the form (3.1) and impose the condition \([Q_1, Q] = 0\). It is satisfied without any restriction on the form of \( Q \). Simplifying by equivalence transformations we obtain

\[
A_{21}^1: \quad Q_1 = \partial_x, \quad Q_2 = \partial_t,
\]

\[
g(x, t) = g_1(x) + ig_2(x),
\]

\[
q(x, t) = q_1(x) + iq_2(x), \quad h(x, t) = h_1(x) + ch_2(x).
\]

\[
A_{21}^2: \quad Q_1 = \partial_x, \quad Q_2 = \partial_x + m(t)\partial_\rho,
\]

\[
g(x, t) = (g_1(t) + ig_2(t))e^{-2\rho(t)},
\]

\[
q(x, t) = (q_1(t) + iq_2(t))e^{-4\rho(t)}, \quad h(x, t) = -i x \dot{m}(t).
\]

\( Q_1 = \partial_x \) commutes with the general element \( Q \) so that we cannot obtain a two-dimensional non-Abelian algebra.

C. Three-dimensional algebras

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Throughout we shall use the nomenclature of the Lie algebra classification given for example in Ref. 19 and list only maximal symmetry algebras.

All three-dimensional solvable algebras have two-dimensional Abelian ideals (nilradicals). We choose a basis \( \{Q_1, Q_2, Q_3\} \) having \( \{Q_1, Q_2\} \) in the ideal and impose the commutation relations

\[
[Q_1, Q_3] = 0, \quad [Q_2, Q_3] = a_1 Q_1 + a_2 Q_2.
\]

If \( a_2 \neq 0 \), by a change of basis one can always set \( a_1 = 0, a_2 = 1 \). This is the case of decomposable algebra \( A_{3, 2} \). If \( a_2 = 0 \), then we can have \( a_1 = 0 \) which is the case of Abelian algebra \( A_{3, 1} \), or we have \( a_1 = 1 \) which corresponds to the nilpotent algebra \( A_{3, 5} \).
Once the form of $Q_3$ has been found from the commutation relations (3.16), the allowed transformations that leave the ideal $\langle Q_1, Q_2 \rangle$ invariant (the residual equivalence group) are then used to simplify $Q_3$.

1. Abelian case

We assume that $Q_1 = \partial \omega$, $Q_2 = \partial x$, $Q_3 = Q$. We already have $[Q_1, Q] = 0$. We then impose the condition

$$[Q_2, Q] = (\alpha + \frac{x}{2} \tau) \partial_x + \tau \partial_x + \hat{A} \rho \partial_\rho + \left( n + \frac{x}{2} \alpha + \frac{x^2}{8} \tau \right) \partial_\omega = 0.$$  \hspace{1cm} (3.17)

We must have $\tau(t) = \tau_0, \alpha(t) = \alpha_0, n(t) = n_0$, all of which are constants. This gives the generator $Q = \alpha_0 \partial_x + \tau_0 \partial_t + A_0 \rho \partial_\rho + n_0 \partial_\omega$. By a change of basis we can make $\tau_0 \rightarrow 0$. If $\alpha_0 = 0$, one of the equations (3.3a)-(3.3d) requires $A_0 = 0$. By assuming that $\alpha_0 \neq 0$ we can rescale $Q$ to have $\alpha_0 \rightarrow 1$ so that $Q = \partial_x + A_0 \rho \partial_\rho + n_0 \partial_\omega$. We solve the determining equations for the coefficients and find after relabeling the constants

$$Q_1 = \partial \omega, \quad Q_2 = \partial_x, \quad Q_3 = \partial_x + A \rho \partial_\rho, \quad g(x, t) = (g_1 + i g_2) \exp(-2Ax), \quad q(x, t) = (q_1 + i q_2) \exp(-4Ax), \quad h(x, t) = h_1 + ih_2,$$

where $\alpha = 2, \quad \beta = 0$. By assuming that $\tau(t) \neq 0$. Then we must have $n(t) = m_0, \alpha(t) = \alpha_0, \tau(t) = \tau_0$. This means we have $Q = \partial_x + m_0 \rho \partial_\rho + \tau_0 \partial_t + A(t) \rho \partial_\rho + n(t) \partial_\omega$. We can rescale $Q$ to have $\tau_0 \rightarrow 1$. A change of basis gives $Q = \partial_x + A(t) \rho \partial_\rho + n(t) \partial_\omega$. When we solve the determining equations, we find the Abelian algebra

$$A_{3,1}^1: \quad Q_1 = \partial \omega, \quad Q_2 = \partial_x + A \rho \partial_\rho, \quad Q_3 = \partial_x + b \rho \partial_\rho, \quad g(x, t) = (g_1 + i g_2) \exp[-2(ax + bt)], \quad q(x, t) = (q_1 + i q_2) \exp[-4(ax + bt)], \quad h(x, t) = 0, \quad a, b, g_1, g_2, q_1, q_2 \in \mathbb{R}.$$  \hspace{1cm} (3.20)

Note that (3.18) is not included in the canonical list of three-dimensional Abelian algebras as it is equivalent to the algebra $A_{3,1}^1$ through the transformation

$$\tilde{x} = x + \tilde{\xi}_0, \quad \tilde{t} = t + \tilde{T}_0, \quad \tilde{\rho} = \rho \exp(h_2 t), \quad \tilde{\omega} = \omega - h_1 t + \tilde{\eta}_0.$$  \hspace{1cm} (3.21)

2. Decomposable case

We start with $A_{3,1}^1$ and let $Q_1 = \partial \omega, Q_2 = \partial_x, Q_3 = Q$. We impose the commutation relation

$$[Q_2, Q] = (\alpha + \frac{x}{2} \tau) \partial_x + \tau \partial_x + \hat{A} \rho \partial_\rho + \left( n + \frac{x}{2} \alpha + \frac{x^2}{8} \tau \right) \partial_\omega = Q_2$$  \hspace{1cm} (3.22)

and find

$$Q = (\alpha_0 + \frac{x}{2}) \partial_x + (t + \tau_0) \partial_t + A_0 \rho \partial_\rho + n_0 \partial_\omega.$$  \hspace{1cm} (3.23)
where all the parameters are constants. We can assume \( \tau_0 = 0 \) up to a change of basis \( Q \rightarrow Q - \tau_0Q \). Applying the allowed transformation (3.10) with the choice \( \epsilon = 1, \xi_0 = 2a_0, T_0 = 0 \) and solving the determining equations we obtain

\[
A_{1,3,2}^1: \quad Q_1 = \partial_\omega, \quad Q_2 = \partial_t, \quad Q_3 = \frac{x}{2} \partial_x + t \partial_t + a \rho \partial_\rho, \\
g(x, t) = \frac{g_1 + ig_2}{x^{2(2a + 1)}}, \quad q(x, t) = \frac{q_1 + iq_2}{x^{2(2a + 1)}}, \quad h(x, t) = \frac{h_1 + ih_2}{x^2},
\]

with some constants \( a, g_1, g_2, q_1, q_2, h_1, h_2 \in \mathbb{R} \).

If we continue with \( A_{2,1}^2 \) for \( Q_1 = \partial_\omega, Q_2 = \partial_x + m(t)\rho \partial_\rho \) and \( Q_3 = Q \), the commutation condition

\[
[Q_2, Q] = \frac{t}{2} \partial_x - \tau m \rho \partial_\rho + \left( \frac{\dot{a}}{4} + \frac{x}{4} \right) \partial_\omega = Q_2
\]

requires \( \tau(t) = 2(t + \tau_0), m(t) = \frac{m_0}{\sqrt{1 + t}}, \alpha(t) = \alpha_0 \), which means we have

\[
Q_2 = \partial_x + \frac{m_0}{\sqrt{1 + t_0}} \rho \partial_\rho, \quad Q = (x + \alpha_0) \partial_x + 2(t + \tau_0) \partial_t + A(t) \rho \partial_\rho + n(t) \partial_\omega.
\]

The allowed transformation (3.13) transforms away \( \alpha_0 \). We solve the determining equations and find that \( A(t) \) and \( n(t) \) must be constants.

\[
A_{1,3,2}^1: \quad Q_1 = \partial_\omega, \quad Q_2 = \partial_t, \quad Q_3 = \frac{x}{2} \partial_x + 2(t + b) \partial_t + c \rho \partial_\rho, \\
g(x, t) = \frac{g_1 + ig_2}{(t + b)^{1+c}} \exp \left( \frac{-4a(b)}{\sqrt{t + b}} \right), \quad q(x, t) = (q_1 + iq_2) \exp(-4ax), \\
h(x, t) = x + \frac{a}{2(t + b)^{3/2}}
\]

with some constants \( a, b, c, g_1, g_2, q_1, q_2 \).

### 3. Nilpotent algebras

We try to realize \( A_{3,5} \) by extending the two-dimensional Abelian algebras \( A_{2,1} = \{Q_1, Q_2\} \) with an element \( Q_3 = Q \) that will satisfy \( [Q_2, Q_3] = Q_1 \). We give the final result skipping the details. \( A_{2,1}^1 \) leads to the nilpotent algebra

\[
A_{1,3,2}^1: \quad Q_1 = \partial_\omega, \quad Q_2 = \partial_t, \quad Q_3 = \partial_x + a \rho \partial_\rho + t \partial_\omega, \\
g(x, t) = (g_1 + ig_2) \exp(-2ax), \quad q(x, t) = (q_1 + iq_2) \exp(-4ax), \\
h(x, t) = x + h_1 + ih_2, \quad a, g_1, g_2, q_1, q_2, h_1, h_2 \in \mathbb{R}.
\]

We find two different realizations from \( A_{2,1}^2 \):

\[
A_{1,3,5}^2: \quad Q_1 = \partial_\omega, \quad Q_2 = \partial_x + a \rho \partial_\rho, \quad Q_3 = 2t \partial_x + t_0 \partial_t + b \rho \partial_\rho + x \partial_\omega, \\
g(x, t) = (g_1 + ig_2) \exp[-2ax + \frac{2}{t_0^0} (at^2 - bt)], \quad q(x, t) = (q_1 + iq_2) \exp[-4ax + \frac{4}{t_0^0} (at^2 - bt)], \quad h(x, t) = 0
\]

with some constants \( t_0 \neq 0, a, b, g_1, g_2, q_1, q_2 \in \mathbb{R} \). If \( t_0 = 0 \) we have

\[
A_{3,5}^3: \quad Q_1 = \partial_\omega, \quad Q_2 = \partial_x, \quad Q_3 = 2t \partial_x + x \partial_\omega, \\
g(x, t) = g_1(t) + ig_2(t), \quad q(x, t) = q_1(t) + iq_2(t), \quad h(x, t) = 0.
\]

This completes the analysis of three-dimensional algebras.
D. Four-dimensional algebras

For the canonical list of four-dimensional Lie algebras we refer the reader to Ref. 19. In the following we skip the details and list only the algebras and the equations.

1. Non-solvable algebras

$A_{3,3}$ trivially extends to the algebra (isomorphic to $\text{gl}(2, \mathbb{R})$)

$A_{3,3} \oplus A_1: \quad Q_1 = \partial_w, \quad Q_2 = \partial_x + t \partial_y - \frac{1}{4} \rho \partial_{\rho}, \quad Q_3 = xt \partial_x + t^2 \partial_y - \frac{1}{2} t \rho \partial_{\rho} + \frac{x^2}{4} \partial_{\omega}, \quad Q_4 = \partial_{\omega},$ \hspace{1cm} (3.30)

\[ g(x, t) = (g_1 + ig_2) \exp(-2bt), \quad q(x, t) = (q_1 + iq_2) \exp(-4bt), \]
\[ h(x, t) = 0, \quad b, g_1, g_2, q_1, q_2 \in \mathbb{R}. \]

This is the semi-direct sum of the three-dimensional Abelian ideal $A_{3,1}$ with the one-dimensional algebra that generates Galilean boosts.

We note that by transformation $\tilde{t} = t, \tilde{x} = x, \tilde{\rho} = \rho \exp(-bt), \tilde{\omega} = \omega$, (3.31) is equivalent to

$A_{4,1}: \quad Q_1 = \partial_w, \quad Q_2 = \partial_x, \quad Q_3 = \partial_y, \quad Q_4 = t \partial_x + \frac{x}{2} \partial_{\omega},$

\[ g(x, t) = (g_1 + ig_2), \quad q(x, t) = (q_1 + iq_2), \]
\[ h(x, t) = ih_2, \quad h_2, g_1, g_2, q_1, q_2 \in \mathbb{R}, \]

which is a more convenient representative of the same equivalence class.

2. Nilpotent algebras

\[ Q_1 = \partial_w, \quad Q_2 = \partial_x, \quad Q_3 = \partial_y, \quad Q_4 = t \partial_x + \frac{x}{2} \partial_{\omega}, \]

\[ g(x, t) = (g_1 + ig_2), \quad q(x, t) = (q_1 + iq_2), \]
\[ h(x, t) = 0, \quad b, g_1, g_2, q_1, q_2 \in \mathbb{R}. \]

3. Indecomposable solvable algebras

There are two types of these algebras constructed from (3.29). The first one is of the form

\[ Q_1 = \partial_w, \quad Q_2 = \partial_x, \quad Q_3 = 2t \partial_x + x \partial_w, \quad Q_4 = x \partial_x + 2t \partial_y + a \rho \partial_{\rho}, \]

\[ g(x, t) = \frac{g_1 + ig_2}{t^{1+a}}, \quad q(x, t) = \frac{q_1 + iq_2}{t^{1+2a}}, \quad h(x, t) = 0. \] \hspace{1cm} (3.33)

Equivalent canonical algebra is obtained by $\tilde{t} = t, \tilde{x} = x, \tilde{\rho} = \rho t^{-(1+a)/2}, \tilde{\omega} = \omega$

$A_{4,8}: \quad Q_1 = \partial_w, \quad Q_2 = \partial_x, \quad Q_3 = 2t \partial_x + x \partial_w, \quad Q_4 = x \partial_x + 2t \partial_y - \rho \partial_{\rho},$

\[ g(x, t) = g_1 + ig_2, \quad q(x, t) = (q_1 + iq_2)t, \quad h(x, t) = \frac{h_2}{t}. \] \hspace{1cm} (3.34)
where $h_2$ is a constant. The second is of the form

$$Q_4 = x t \partial_x + (1 + t^2)\partial_t - \frac{1}{2}(b + t)p\partial_p + \frac{x^2}{4}\partial_\omega,$$

(3.35)

$$g(x, t) = \frac{g_1 + ig_2}{\sqrt{1 + t^2}} \exp(b \arctan t),$$

$$q(x, t) = (q_1 + iq_2)\exp(2b \arctan t), \quad h(x, t) = 0.$$

Again it will be more convenient to use the following equivalent algebra

$$A_{4.9} : \quad Q_4 = x t \partial_x + (1 + t^2)\partial_t - tp\partial_p + \frac{x^2}{4}\partial_\omega,$$

(3.36)

$$g(x, t) = (g_1 + ig_2), \quad q(x, t) = (q_1 + iq_2)(1 + t^2), \quad h(x, t) = i \frac{t + h_2}{2(1 + t^2)},$$

which is achieved by $\ddot{t} = t, \ddot{x} = x, \ddot{\rho} = (1 + t^2)^{-1/4} \exp(b \arctan t), \ddot{\omega} = \omega.$

### E. Five- and six-dimensional algebras

As the equations invariant under algebras up to dimension four contain only parameters it is reasonable to leave the above strategy and identify the maximal symmetry algebras by using the direct Lie algorithm. So we solve the determining equations for the classes (3.30), (3.32), (3.34), (3.36) to find any further possible extensions. This is easy to do. We skip all the details and sum up our results as theorems.

**Theorem 1:** The symmetry group of the genuine ($g$ and $q$ not both zero) variable coefficient CQNLS equation can be at most four-dimensional. There are precisely four inequivalent classes of equations given by (3.30), (3.32), (3.34), (3.36).

In the cubic case, $A_{4.1}$ with $h_2 = 0, A_{4.8}$ with $h_2 = \{0, 1/2\}$ extend to a five-dimensional algebra. In the quintic case, $A_{3.3} \oplus A_1$ with $g = h = 0, A_{4.1}$ with $h_2 = 0, A_{4.8}$ with $h_2 = 1/4$, and $A_{4.9}$ with $h_2 = 0$ extend to a six-dimensional algebra.

**Theorem 2:** Any variable coefficient CQNLS equation with a five- or six-dimensional symmetry group can be transformed into the standard cubic equation

$$Q_1 = \partial_x, \quad Q_2 = \partial_t, \quad Q_3 = \partial_\omega,$$

(3.37)

$$Q_4 = t \partial_x + \frac{x}{2} \partial_\omega, \quad Q_5 = \frac{x}{2} \partial_x + t \partial_\omega - \frac{1}{2} p \partial_p,$$

$$g(x, t) = g_1 + ig_2, \quad q(x, t) = 0, \quad h(x, t) = 0, \quad g_1, g_2 \in \mathbb{R},$$

or quintic equation

$$Q_1 = \partial_x, \quad Q_2 = \frac{x}{2} \partial_x + t \partial_\omega - \frac{1}{4} p \partial_p,$$

(3.38)

$$Q_3 = x t \partial_x + t^2 \partial_t - \frac{1}{2} t \rho \partial_\rho + \frac{x^2}{4}\partial_\omega,$$

$$Q_4 = \partial_\omega, \quad Q_5 = t \partial_\omega + \frac{x}{2} \partial_\omega, \quad Q_6 = \partial_x,$$

$$g(x, t) = 0, \quad q(x, t) = q_1 + iq_2, \quad h(x, t) = 0, \quad q_1, q_2 \in \mathbb{R},$$

respectively. The symmetry algebra is isomorphic to the one-dimensional Galilei similitude algebra $\mathfrak{g}_s(1)$ in the first case, and to the one-dimensional Schrödinger $\mathfrak{sch}(1)$ algebra in the second case.
IV. TRANSFORMATION TO THE STANDARD CQNLS EQUATION

We can take advantage of the equivalence transformations to establish the conditions for the transformability of Eq. (1.1) with $f = 1$ to the constant-coefficient cubic-quintic equation, namely

$$\tilde{f} = 1, \quad \tilde{g} = a_1 + ia_2 \neq 0, \quad \tilde{q} = b_1 + ib_2 \neq 0, \quad \tilde{k} = \tilde{h} = 0,$$

(4.1)

where $a_1, a_2, b_1, b_2$ are real constants.

We already know that $\tilde{f} = 1$ constrains $X$ to

$$X(x, t) = \epsilon \sqrt{T} x + \xi(t).$$

(4.2)

(2.3b) and (2.3c) imply that

$$\tilde{g} = (g_1 + ig_2) \frac{R^2}{T} = a_1 + ia_2,$$

(4.3a)

$$\tilde{q} = (q_1 + iq_2) \frac{R^4}{T} = b_1 + ib_2.$$

(4.3b)

Let us assume that $g_1(x, t) \neq 0$, which requires $a_1 \neq 0$. From (4.3) we have the requirements

$$g(x, t) = g_1(x, t)(1 + i \frac{a_2}{a_1}), \quad q(x, t) = \frac{b_1 + ib_2 g_1^2(x, t)}{a_1^2} \gamma(t).$$

(4.4)

We further have

$$T(t) = \int \gamma(t) dt, \quad R(x, t) = \left( \frac{g_1 \gamma(t)}{g_1(x, t)} \right)^{1/2}.$$

(4.5)

The condition $\tilde{k} = 0$ gives

$$X_x(2 \frac{R_x}{R} + k_1) + i(X_t + 2X_x \theta_x + k_2 X_x) = 0$$

(4.6)

leading to

$$k_1(x, t) = \frac{g_1 x}{g_1},$$

(4.7)

$$\theta(x, t) = -\frac{\dot{\gamma}}{8y} x^2 - \frac{\dot{\xi}}{2 \epsilon \sqrt{\gamma}} x - \frac{1}{2} \int k_2(x, t) dx + \eta(t).$$

(4.8)

Having determined the transformations, we can construct the corresponding admissible potentials from the condition $\tilde{h} = 0$ as

$$h_1(x, t) = \left( \frac{3}{16} \left( \frac{\dot{\gamma}}{\gamma} \right)^2 - \frac{\dot{\xi}}{8y} \right) x^2 + \frac{1}{2 \epsilon} \left( \frac{\dot{\xi} \gamma - \dot{\gamma} \xi}{\gamma^{3/2}} \right) x$$

$$+ \frac{g_1 x}{2 g_1} - \frac{g_1^2}{4 g_1^2} - \frac{k^2}{4} - \frac{1}{2} \int k_{2, t} dx + \frac{\dot{k}^2}{4 \gamma} + \dot{\eta},$$

(4.9a)

$$h_2(x, t) = \frac{g_1 x}{2 g_1} + \frac{k_2 g_1 x}{2 g_1} + \frac{1}{2} k_{2, x} - \frac{\dot{\gamma}}{4 \gamma}.$$ 

(4.9b)

Summarizing, transformation of (1.1) with coefficients $f = 1$ and those given by (4.4), (4.7), and (4.9) to the standard cubic-quintic equation can be made possible by the special equivalence group given by $X, T, R, \theta$ in (4.2), (4.5), and (4.8). One can use these results to obtain solutions from those of the constant coefficient equation extensively studied in literature, for example, Ref. 20.
Remark 2: Radial case:
For \( k(x, t) = (n - 1)x \), from (4.7) we must have \( g_1(x, t) \) as in (4.5) with an arbitrary \( G \).
Provided the condition (4.4) holds, the admissible potential has the form

\[
h_1(x, t) = \left( \frac{3}{16} \frac{\dot{\gamma}}{\gamma^2} - \frac{\ddot{\gamma}}{8\gamma} \right) x^2 + \frac{1}{2e} \left( \frac{\ddot{\gamma}}{\gamma^3/2} \right) x
+ \frac{(n - 1)(n - 3)}{4x^2} + \frac{1}{4} \frac{\dot{\gamma}^2}{\gamma} + \ddot{\gamma},
\]
(4.10a)

and the corresponding transformations are given by (4.2), (4.5), (4.8).

Remark 3: The coefficient of the quadratic term in \( h_1 \) can be identified with a multiple of the Schwarzian derivative of \( \dot{T}(t) \) in the form \(-\{T, t\}/8 \). This implies that if the potential does not contain a quadratic term, then the time transformation \( T \) appears to correspond to the linear fractional (or Möbius) transformations in \( t \) depending on three parameters. However, \( \gamma \) in \( q \) of (4.4) is fixed by the relation \( \gamma = \dot{T}(t) \) in this case.

Remark 4: In the cubic-quintic case, equations corresponding to the four-dimensional algebras \( A_{3,3} \oplus A_1, A_{4,8} \), and \( A_{4,9} \) fail to satisfy all of the conditions (4.4), (4.7), (4.9) and therefore cannot be transformed to their constant coefficient analogues. These conditions hold for \( A_{4,1} \) only when \( h_2 = 0 \), which is already the symmetry algebra of the constant coefficient CQNLS equation. This implies that the transformable classes fall outside of these equations.

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