An Algorithm for Finding the Periodic Potential of the Three-Dimensional Schrödinger Operator from the Spectral Invariants

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Abstract

In this paper, we investigate the three-dimensional Schrödinger operator with a periodic, relative to a lattice Ω of $\mathbb{R}^3$, potential q. A special class V of the periodic potentials is constructed, which is easily and constructively determined from the spectral invariants. First, we give an algorithm for the unique determination of the potential $q \in V$ of the three-dimensional Schrödinger operator from the spectral invariants that were determined constructively from the given Bloch eigenvalues. Then we consider the stability of the algorithm with respect to the spectral invariants and Bloch eigenvalues. Finally, we prove that there are no other periodic potentials in the set of large class of functions whose Bloch eigenvalues coincides with the Bloch eigenvalues of $q \in V$.

Keywords: Schrödinger operator, spectral invariants, inverse problem.

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1 Introduction

Let $L(q)$ be the Schrödinger operator

$$L(q) = -\Delta + q(x),\ \ x \in \mathbb{R}^d \tag{1}$$

with a periodic, relative to a lattice Ω, potential $q(x)$. The operator $L(q)$ describes the motion of a particle in bulk matter. Therefore, for physical applications, it is important to have a detailed analysis of the spectral properties of $L(q)$. Let $F =: \mathbb{R}^d/\Omega$ be a fundamental domain of Ω and $L_t(q)$ be the operator generated in $L_2(F)$ by (1) and the conditions:

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \ \forall\omega \in \Omega,$$

where $t \in F^\star =: \mathbb{R}^d/\Gamma$, Γ is the lattice dual to Ω, that is, Γ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $\langle \gamma, \omega\rangle \in 2\pi\mathbb{Z}$ for all $\omega \in \Omega$, and $\langle .,.\rangle$ is the inner product in $\mathbb{R}^d$. It is well-known that (see [1]) the spectrum of $L(q)$ is the union of the spectra of $L_t(q)$ for $t \in F^*$. The eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq ...$ of $L_t(q)$ are called the Bloch eigenvalues of $L(q)$. These eigenvalues define the continuous functions $\Lambda_1(t)$, $\Lambda_2(t), ...$ of t that are called the band functions of $L(q)$. The intervals $\{\Lambda_1(t) : t \in F^\star\}$, $\{\Lambda_2(t) : t \in F^\star\}, ...$ are the bands of the spectrum of $L(q)$ and the spaces (if exists) $\Delta_1, \Delta_2, ...$ between two neighboring bands are the gaps in the spectrum. Without loss of generality, we assume that the measure of F is 1.

In this paper we determine constructively and uniquely, modulo the following inversion and translations

$$q(x) \to q(-x),\ \ q(x) \to q(x+\tau),\ \ \tau \in \mathbb{R}^d, \tag{2}$$

the potential $q(x)$ of the three- dimensional ($d=3$) Schrödinder operator $L(q)$ from the given spectral invariants , when $q(x)$ has the form

$$q(x)=\sum_{a\in Q(1,1,1)} z(a)e^{i\langle a,x\rangle}, \tag{3}$$

where

$$z(a)=:(q(x),e^{i\langle a,x\rangle})\neq 0,\ \forall a\in Q(1,1,1), \tag{4}$$

$$Q(1,1,1)=:\{n\gamma_1+m\gamma_2+s\gamma_3:|n|\leq 1,\ |m|\leq 1,\ |s|\leq 1\}\backslash\{(0,0,0)\}, \tag{5}$$

$(.,.)$ is the inner product in $L_2(F)$ and $\{\gamma_1,\gamma_2,\gamma_3\}$ is a basis of Γ satisfying

$$\langle\gamma_i,\gamma_j\rangle\neq 0,\ \langle\gamma_i+\gamma_j,\gamma_k\rangle\neq 0,\ \ |\gamma_i|\neq|\gamma_j|,\ \langle\gamma_i+\gamma_j+\gamma_k,\gamma_i-\gamma_j-\gamma_k\rangle\neq 0 \tag{6}$$

for all different indices i,j,k. Note that every lattice has a basis satisfying (6) (see Proposition 2 in Section 2) and the potential q can be uniquely determined only by fixing the inversion and translations (2), since the operators $L(q(x)),\ L(q(-x)),\ L(q(x+\tau))$ have the same band functions.

The inverse problem of the one-dimensional Schrödinger operator, that is, the Hill operator, denoted by $H(q)$, and the multidimensional Schrödinger operator $L(q)$ are absolutely different. Inverse spectral theory for the Hill operator has a long history and there exist many books and papers about it (see, for example, [6] and [7]). In order to determine the potential q, where $q(x+\pi)=q(x)$, of the Hill operator, in addition to the given band functions $\Lambda_1(t)$, $\Lambda_2(t),...$, one needs to know the eigenvalues $\lambda_1,\lambda_2,...$ of the Dirichlet boundary value problem and the signs of the numbers $u_-(\sqrt{\lambda_1})$, $u_-(\sqrt{\lambda_2}),...$, where $u_-(\lambda)=c(\lambda,\pi)-s^{'}(\lambda,\pi)$ and $c(\lambda,x)$, $s(\lambda,x)$ are the solutions of the Hill equation

$$-y^{''}(x)+q(x)y(x)=\lambda^2y(x)$$

satisfying $c(\lambda,0)=s^{'}(\lambda,0)=1,\ c^{'}(\lambda,0)=s(\lambda,0)=0$ (see [7], Chap.3, Sec. 4). In other words, the potential q of the Hill operator can not be determined uniquely from the given band functions, since if the band functions $\Lambda_1(t),\ \Lambda_2(t),...$ of $H(q)$ are given, then for every choice of the numbers $\lambda_1,\lambda_2,...$ from the gaps $\Delta_1,\Delta_2,...$ of the spectrum of the Hill operator and for every choice of the signs of the numbers $u_-(\lambda_1),\ u_-(\lambda_2),...$, there exist a potential q having $\Lambda_1(t),\ \Lambda_2(t),...$ as a band functions and $\lambda_1,\lambda_2,...$ as the Dirichlet eigenvalues. In spite of this, it is possible to determine uniquely (modulo (2)) the potential q of the multidimensional Schrödinger operator $L(q)$ from only the given band functions for a certain class of potential. Because, in the case $d>1$ the band functions give more informations. Namely, the band functions give the spectral invariants that have no meaning in the case $d=1$. We solve the inverse problem by these spectral invariants. We will discuss this in the end of the introduction.

The inverse problem for the multidimensional Schrödinger operator $L(q)$ for the first time is investigated by Eskin, G., Ralston, J., Trubowitz, E. in the papers [2,3]. In [2] it is proved the following result:

Assume that the lattice Ω of $\mathbb{R}^d$ is such that, for $\omega,\omega^{'}\in\Omega,|\omega^{'}|=|\omega|$ implies $\omega^{'}=\pm\omega$. If $q(x)$ and $\widetilde{q}(x)$ are real analytic, then the equality

$$Spec(L_0(q))=Spec(L_0(\widetilde{q})) \tag{7}$$

implies the equalities

$$Spec(L_t(q))=Spec(L_t(\widetilde{q})) \tag{8}$$

for all $t\in\mathbb{R}^d$, where $Spec(L_t(q))$ is the spectrum of the operator $L_t(q)$ and $L_0(q)$ is the *operator* $L_t(q)$ *when* $t = (0, 0, ..., 0)$.

In [3] it is proved the following result for the two-dimensional Schrödinger operator $L(q)$:

For $\Omega \subset \mathbb{R}^2$ *satisfying the condition: if* $| \omega^{'} |=| \omega |$ *for* $\omega, \omega^{'} \in \Omega$, *then* $\omega^{'} = \pm\omega$; *there is a set* $\{M_\alpha\}$ *of manifolds of potentials such that*

a) $\{M_\alpha : \alpha \in [0, 1]\}$ *is dense in the set of smooth periodic potentials in the* C^∞*-topology,*

b) for each α *there is a dense open set* $Q_\alpha \subset M_\alpha$ *such that for* $q \in Q_\alpha$ *the set of real analytic* $\widetilde{q}$ *satisfying (7)and the set of* $\widetilde{q} \in C^6(F)$ *satisfying (8) for all* $t \in \mathbb{R}^2$ *are finite modulo translations in (2).*

Eskin, G. [4] extend the results of the papers [2,3] to the case of two-dimensional Schrödinger operator

$$H = (i\nabla + A(x))^2 + V(x),\ x \in \mathbb{R}^2$$

with periodic magnetic potential $A(x) = (A_1(x), A_2(x))$ and electric potential $V(x)$. The proof of the results of the papers [2-4] is not constructive and does not seem to give any idea about possibility to construct explicitly a periodic potential.

In the paper [11] we constructed a set D of trigonometric polynomials of the form

$$q(x) = \sum_{a \in Q(N,M,S)} z(a)e^{i\langle a,x\rangle}, \tag{9}$$

where $Q(N, M, S) =: \{(n, m, s) : |n| \le N,\ |m| \le M,\ |s| \le S\}\backslash\{(0, 0, 0)\}$ is a subset of the lattice $\mathbb{Z}^3$, satisfying the following conditions:

$z(n, m, s) \neq 0$ for $(n, m, s) \in B(N, M, S) \cup C(\sqrt{N})$ and

$z(n, m, s) = 0$ for $(n, m, s) \in (Q(N, M, S))\backslash(C(\sqrt{N}) \cup B(N, M, S))$, where

$$B(N, M, S) = \{(n, m, s) \in Q(N, M, S) : nms(|n| - N)(|m| - M)(|s| - S) = 0\},$$

$$C(\sqrt{N}) = \left\{(n, m, s) : 0 <| n |< \tfrac{1}{2}\sqrt{N}, 0 <| m |< \tfrac{1}{2}\sqrt{N}, 0 <| s |< \tfrac{1}{2}\sqrt{N}\right\}$$

and N, M, S are large prime numbers satisfying $S > 2M$, $M > 2N$, $N \gg 1$. Then we proved that: D is dense in $W_2^s(\mathbb{R}^3/\Omega)$, where $s > 3$, in the C^∞- topology and any element q of the set D can be determined constructively and uniquely, modulo inversion and translations (2), from the given Bloch eigenvalues.

Thus, in the papers [2-4] and in our papers (in [11] and in this paper) the different aspects of the inverse problem are investigated by absolutely different methods. It follows from (3) and from the conditions on (9) that the intersection of the set of potentials investigated in [11] and in this paper is empty set. In this paper and in [11] we determine constructively the potential q of the three-dimensional Schrödinger operator $L(q)$ from the spectral invariants that were determined constructively in [10] from the given band functions. As a result, we determine constructively the potential from the given band functions. Actually, we do not only show how to construct a periodic potential q with the desired properties but even present an algorithm for the construction. Moreover, in this paper we investigate the stability of the algorithm and prove some uniqueness theorems which were not done in [11].

To describe the brief scheme of this paper, we begin by recalling the definition of some well-known concepts and the invariants obtained in [10] that will be used here. An element a of the lattice Γ is said to be a visible element of Γ if a is an element of Γ of the minimal norm belonging to the line $a\mathbb{R}$. Denote by S the set of all visible elements of Γ. Clearly,

$$q(x) = \frac{1}{2}\sum_{a \in S} q^a(x), \tag{10}$$

where

$$q^a(x) = \sum_{n \in \mathbb{Z}} z(na)e^{in\langle a,x\rangle}. \tag{11}$$

The function $q^a(x)$ defined in (11) are known as directional potential of (10) corresponding to the visible element a. In Proposition 1 of Section 2 we prove that every element a of $Q(1,1,1)$ is a visible element of Γ and the directional potential (11) of (3) has the form

$$q^a(x) = z(a)e^{i\langle a,x\rangle} + z(-a)e^{-i\langle a,x\rangle}. \tag{12}$$

Let a be a visible element of Γ, Ω_a be the sublattice $\{\omega \in \Omega : \langle \omega, a\rangle = 0\}$ of Ω in the hyperplane $H_a = \{x \in \mathbb{R}^3 : \langle x, a\rangle = 0\}$ and

$$\Gamma_a =: \{\gamma \in H_a : \langle \gamma, \omega\rangle \in 2\pi\mathbb{Z},\ \forall \omega \in \Omega_a\} \tag{13}$$

be the lattice dual to Ω_a. Let β be a visible element of Γ_a and $P(a,\beta)$ be the plane containing a, β and the origin. Define a function $q_{a,\beta}(x)$ by

$$q_{a,\beta}(x) = \sum_{c\in(P(a,\beta)\cap\Gamma)\backslash a\mathbb{R}} \frac{c}{\langle \beta, c\rangle} z(c)e^{i\langle c,x\rangle}. \tag{14}$$

In the paper [10] we constructively determined the following spectral invariants

$$I(a) = \int_F |q^a(x)|^2\, dx, \tag{15}$$

$$I_1(a,\beta) = \int_F |q_{a,\beta}(x)|^2\, q^a(x)dx \tag{16}$$

from the asymptotic formulas for the band functions of $L(q)$ obtained in [8,9], where $q^a(x)$ is the directional potential (11). Moreover, in [10] we constructively determined the invariant

$$I_2(a,\beta) = \int_F |q_{a,\beta}(x)|^2 (z^2(a)e^{i2\langle a,x\rangle} + z^2(-a)e^{-i2\langle a,x\rangle})dx \tag{17}$$

when $q^a(x)$ has the form (12). Since all the directional potential of (3) have the form (12) (see Proposition 1), we have the invariants (15)-(17) for all $a \in Q(1,1,1)$.

In Section 2 we describe the invariants (15)-(17) for (3).

In Section 3 fixing the inversion and translations (2), we give an algorithm for the unique determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the invariants (15)-(17). Since the invariants (16) and (17) do not exist in the case $d = 1$, we do not use the investigations of the inverse problem for the one dimensional Schrödinger operator $H(q)$. For this reason, we do not discuss a great number of papers about the inverse problem of the Hill operator.

In Section 4 we study the stability of the algorithm with respect to errors both in the invariants (15)-(17) and in the Bloch eigenvalues. Note that we determine constructively the potential from the band functions in two steps. At the first step we determined the invariants from the band functions in the paper [10]. At the second step, which is given in Section 3, we find the potential from the invariants. In Section 4 we consider the stability of the problems studied in the both steps. First, using the asymptotic formulas obtained in [10], we write down explicitly the asymptotic expression of the invariants (15)-(17) in terms of the band functions and consider the stability of the invariants with respect to the errors in the Bloch eigenvalues (Theorem 5). Then we prove the stability of the algorithm given in Section 3 with respect to the errors in the invariants (Theorem 6).

In Section 5 we prove some uniqueness theorems. First, we prove a theorem about Hill operator $H(p)$ when $p(x)$ is a trigonometric polynomial (see Theorem 7). Then we construct a set W of all periodic functions $q(x)$ whose directional potentials (see (10), (11)) $q^a(x)$ for all $a \in S \backslash \{\gamma_1, \gamma_2, \gamma_3\}$ are arbitrary continuously differentiable functions, where S is the set of all visible elements of Γ, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying (6), and the directional potentials $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$, $q^{\gamma 3}(x)$ satisfy some conditions. At the end we prove that if q is of the form (3), $\widetilde{q} \in W$ and the band functions of $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ is equal to q modulo inversion and translations (2) (see Theorem 8).

2 On the spectral invariants (15)-(17)

To describe the invariant (15) let us prove the following proposition.

Proposition 1 *Every element a of the set $Q(1,1,1)$, defined in (5), is a visible element of Γ and the corresponding directional potential (11) has the form (12).*

Proof. Let a be element of $Q(1,1,1)$. By the definition of $Q(1,1,1)$

$$a = n\gamma_1 + m\gamma_2 + s\gamma_3, \ |n| \leq 1, \ |m| \leq 1, \ |s| \leq 1, \ a \neq 0. \tag{18}$$

If a is not a visible element of Γ, then there exists a visible element b of Γ such that $a = kb$ for some integer $k > 1$. This with (18) implies that

$$b = \tfrac{1}{k}(n\gamma_1 + m\gamma_2 + s\gamma_3). \tag{19}$$

Since $b \in \Gamma$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ we have $b = n_1\gamma_1 + m_1\gamma_2 + s_1\gamma_3$, where n_1, m_1, s_1 are integers. Combining this with (19) and taking into account the linearly independence of the vectors $\gamma_1, \gamma_2, \gamma_3$, we get

$$(n_1 - \tfrac{n}{k})\gamma_1 + (m_1 - \tfrac{m}{k})\gamma_2 + (s_1 - \tfrac{s}{k})\gamma_3 = 0, \text{ and } n_1 - \tfrac{n}{k} = \ m_1 - \tfrac{m}{k} = \ s_1 - \tfrac{s}{k} = 0.$$

This is impossible, since $|n| \leq 1, \ |m| \leq 1, \ |s| \leq 1$, at least one of the numbers $n, \ m, \ s$ is not zero (see (18)), $k > 1$ and the numbers n_1, m_1, s_1 are integers. This contradiction shows that any element a of $Q(1,1,1)$ is a visible element of Γ. Therefore, it follows from the definition of $Q(1,1,1)$ (see (5)) that the line $a\mathbb{R}$ contains only two elements a and $-a$ of the set $Q(1,1,1)$. This means that the directional potential (11) has the form (12) ■

By Proposition 1 the invariant (15) for the potential (3) has the form

$$I(a) = |z(a)|^2, \ \forall a \in Q(1,1,1), \tag{20}$$

that is, we determine the absolute value of $z(a)$ for all $a \in Q(1,1,1)$.

To investigate the invariants (16) and (17), we use the conditions in (6). Therefore, first, let us consider these conditions.

Proposition 2 *Any lattice Γ has a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfying (6). In particular, if*

$$\Gamma = \{(na, mb, sc) : n, m, s \in \mathbb{Z}\}, \tag{21}$$

where $a, b, c \in \mathbb{R} \backslash \{0\}$, then at least one of the bases $\{(a,0,0), (a,b,0), (a,b,c)\}$ and $\{(-a,0,0), (a,b,0), (a,b,c)\}$ of Γ satisfies (6).

Proof. Suppose that a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of Γ does not satisfy (6). Define $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ by

$$\widetilde{\gamma}_1 = \gamma_1, \ \widetilde{\gamma}_2 = n\gamma_1 + \gamma_2, \ \widetilde{\gamma}_3 = m\gamma_1 + s\gamma_2 + \gamma_3,$$

where n, m, s are integers. Since $\gamma_1 = \widetilde{\gamma}_1$, $\gamma_2 = \widetilde{\gamma}_2 - n\widetilde{\gamma}_1$, $\gamma_3 = \widetilde{\gamma}_3 - m\widetilde{\gamma}_1 - s(\widetilde{\gamma}_2 - n\widetilde{\gamma}_1)$, the triple $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ is also basis of Γ. In (6) replacing $\{\gamma_1, \gamma_2, \gamma_3\}$ by $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$, we obtain

12 inequalities with respect to n, m and s. Since n, m and s are arbitrary integers one can readily see that there exists n, m and s for which these inequalities hold. For example, let

$$\widetilde{\gamma}_1 = \gamma_1,\ \widetilde{\gamma}_2 = n\gamma_1 + \gamma_2,\ \widetilde{\gamma}_3 = n^2\gamma_1 + \gamma_3, \tag{22}$$

where n is a large positive number, that is, $n \gg 1$. Then it follows from (22) that

$$\langle \widetilde{\gamma}_i,\ \widetilde{\gamma}_j \rangle \gg 1,\ \ \langle \widetilde{\gamma}_i + \widetilde{\gamma}_j, \widetilde{\gamma}_j \rangle \gg 1,\ \forall i \neq j,$$

that is, the first and second inequalities in (6) hold. Besides, by (22), we have

$$\mid \widetilde{\gamma}_1 \mid^2 \sim 1,\ \ \mid \widetilde{\gamma}_2 \mid^2 \sim n^2,\ \ \mid \widetilde{\gamma}_3 \mid^2 \sim n^4,\ \ \langle \widetilde{\gamma}_i,\ \widetilde{\gamma}_j \rangle = O(n^3), \tag{23}$$

where $a_n \sim b_n$ means that there exist positive constants c_1 and c_2 such that $c_1 \mid b_n \mid < \mid a_n \mid < c_2 \mid b_n \mid$, for $n = 1, 2, \ldots$. The third inequality of (6) holds due to (23). By (23) the term $\pm \mid \widetilde{\gamma}_3 \mid^2$ in the fourth inequality of (6) can not be canceled by the other terms of this inequality. Thus, we proved that any lattice Γ has a basis $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ satisfying (6).

Note that, for the given lattice, one can easily find the basis satisfying (6). For example, in the case (21), one can readily see that the basis $\{(a,0,0),(a,b,0),(a,b,c)\}$ satisfies (6) if $c^2 \neq 3a^2$ and the basis $\{(-a,0,0),(a,b,0),(a,b,c)\}$ satisfies (6) if $c^2 \neq a^2$. Thus at least one of the bases $\{(a,0,0),(a,b,0),(a,b,c)\}$ and $\{(-a,0,0),(a,b,0),(a,b,c)\}$ satisfies (6) ■

Now to describe the invariants (16) and (17) for (3) let us introduce some notations. If $b \in (\Gamma \cap P(a,\beta))\backslash a\mathbb{R}$, then the plane $P(a,\beta)$ coincides with the plane $P(a,b)$. Moreover, every vector $b \in (P(a,\beta) \cap \Gamma)\backslash a\mathbb{R}$ has an orthogonal decomposition (see (20) in [8])

$$b = s\beta + \mu a, \tag{24}$$

where s is a nonzero integer, β is a visible element of Γ_a (see (13)) and μ is a real number. Therefore, for every plane $P(a,b)$, where $b \in \Gamma$, there exists a plane $P(a,\beta)$, where β is defined by (24), coinciding with $P(a,b)$. For every pair $\{a,b\}$, where a is visible element of Γ and $b \in \Gamma$, we redenote by $I_1(a,b)$ and $I_2(a,b)$ the invariants $I_1(a,\beta)$ and $I_2(a,\beta)$ defined in (16) and (17) respectively, where β is a visible element of Γ_a defined by (24).

Theorem 1 *The following equalities for the invariant (16) hold:*

$$I_1(\gamma_i + \gamma_j, \gamma_i) = A_1(\gamma_i + \gamma_j, \gamma_i)\,\mathrm{Re}(z(-\gamma_i - \gamma_j)z(\gamma_j)z(\gamma_i)), \tag{25}$$

$$I_1(\gamma_i - \gamma_j, \gamma_i) = A_1(\gamma_i - \gamma_j, \gamma_i)\,\mathrm{Re}(z(-\gamma_i + \gamma_j)z(-\gamma_j)z(\gamma_i)), \tag{26}$$

$$I_1(\gamma, \gamma_i) = A_1(\gamma, \gamma_i)\,\mathrm{Re}(z(-\gamma)z(\gamma - \gamma_i)z(\gamma_i)), \tag{27}$$

$$I_1(2\gamma_i - \gamma, \gamma_i) = A_1(2\gamma_i - \gamma, \gamma_i)\,\mathrm{Re}(z(\gamma - 2\gamma_i)z(\gamma_i - \gamma)z(\gamma_i)), \tag{28}$$

where $A_1(\gamma_i \pm \gamma_j, \gamma_i)$, $A_1(\gamma, \gamma_i)$, $A_1(2\gamma_i - \gamma, \gamma_i)$ are nonzero numbers defined by

$$A_1(a,b) = 2\left((\langle b,\beta\rangle)^{-2} + (\langle a-b,\beta\rangle)^{-2}\right)\langle a-b, b\rangle\,, \tag{29}$$

$\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying (6), $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $\mathrm{Re}(z)$ is the real part of z.

Proof. If the potential $q(x)$ has the form (3), then (14) becomes

$$q_{a,\beta}(x) = \sum_{c \in (P(a,\beta) \cap Q)\backslash a\mathbb{R}} \frac{c}{\langle \beta, c\rangle} z(c) e^{i\langle c,x\rangle}, \tag{30}$$

where, for brevity, $Q(1,1,1)$ is denoted by Q. Using this and (11) in (16) and taking into account that the invariant $I_1(a,\beta)$ defined by (16) is redenoted by $I_1(a,b)$, we get

$$I_1(a,b) = \Sigma_1 + \Sigma_2, \tag{31}$$

where

$$\Sigma_1 = \sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c+a\rangle}{\langle c,\beta\rangle\,\langle c+a,\beta\rangle} z(c)z(-a-c)z(a),$$

$$\Sigma_2 = \sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c-a\rangle}{\langle c,\beta\rangle\,\langle c-a,\beta\rangle} z(c)z(a-c)z(-a)$$

and β is a visible element of Γ_a defined by (24). Since $Q(1,1,1)$ is symmetric with respect to the origin, the substitution $\widetilde{c} = -c$ in Σ_1 does not change Σ_1. Using this substitution in Σ_1 and then taking into account that $z(-b) = \overline{z(b)}$, $\langle a,\beta\rangle = 0$, we obtain $\Sigma_1 = \overline{\Sigma_2}$. This with (31) gives

$$I_1(a,b) = 2\,\mathrm{Re}\left(z(-a)\left(\sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle a-c, c\rangle}{(\langle c,\beta\rangle)^2} z(a-c)z(c)\right)\right). \tag{32}$$

Since a, β, $(0,0,0)$ belong to the plane $P(a,b)$ and β orthogonal to the line $a\mathbb{R}$, we have

$$\langle c,\beta\rangle \neq 0,\ \forall c \in (P(a,b)\cap Q)\backslash a\mathbb{R}. \tag{33}$$

Now using (32) we obtain the invariants (25) and (26) as follows. First let us consider (25). Let $a = \gamma_i + \gamma_j$ and $b = \gamma_i$. Then

$$(P(a,b)\cap Q)\backslash a\mathbb{R} = \{\pm\gamma_i,\ \ \pm\gamma_j,\ \ \pm(\gamma_i - \gamma_j)\}.$$

One the other hand, if $c \in \{-\gamma_i, -\gamma_j, \pm(\gamma_i-\gamma_j)\}$, then $a-c \notin Q$. Therefore, the summation in the formula (32) for the case $a = \gamma_i+\gamma_j$, $b = \gamma_i$ is taken over $c \in \{\gamma_i,\ \gamma_j\}$ and hence (25) holds. It follows from (33) and from the first inequality in (6) that $A_1(\gamma_i+\gamma_j,\gamma_i) \neq 0$.

Replacing $a = \gamma_j$ by $-\gamma_j$ and arguing as in the proof of (25), we get (26).

Now let us consider (27). Let $a = \gamma = \gamma_1+\gamma_2+\gamma_3$ and $b = \gamma_1$. Then

$$(P(a,b)\cap Q)\backslash a\mathbb{R} = \{\pm\gamma_1,\ \ \pm(\gamma_2+\gamma_3)\}.$$

One the other hand, if $c = -\gamma_1$, or $c = -\gamma_2-\gamma_3$, then $a-c \notin Q$. Therefore, the summation in the formula (32) for this case is taken over $c \in \{\gamma_1,\ \gamma_2+\gamma_3\}$ and hence (27) holds for $i = 1$. In the same way, we obtain (27) for $i = 2, 3$.

Now let us consider (28). Let $a = 2\gamma_i - \gamma$ and $b = \gamma_i$. Then

$$(P(a,b)\cap Q)\backslash a\mathbb{R} = \{\pm\gamma_i,\ \ \pm(\gamma_i-\gamma)\}.$$

One the other hand, if $c = -\gamma_i$, or $c = \gamma-\gamma_i$, then $a-c \notin Q$. Therefore, the summation in the formula (32) for this case is taken over $c \in \{\gamma_i, \gamma_i-\gamma\}$ and hence (28) holds. Since $\gamma_i - \gamma = -(\gamma_j+\gamma_k)$, it follows from the second inequality in (6) that $A_1(2\gamma_i-\gamma,\gamma_i) \neq 0$. ■

Theorem 2 *The following equalities for the invariant (17) hold:*

$$I_2(\gamma_i,\gamma_j) = A_2(\gamma_i,\gamma_j)\,\mathrm{Re}(z^2(-\gamma_i)z(\gamma_i+\gamma_j)z(\gamma_i-\gamma_j)), \tag{34}$$

$$I_2(\gamma_i,\gamma-\gamma_i) = A_2(\gamma_i,\gamma-\gamma_i)\,\mathrm{Re}(z^2(-\gamma_i)z(\gamma)z(2\gamma_i-\gamma)), \tag{35}$$

where $A_2(\gamma_i,\gamma_j)$, $A_2(\gamma_i,\gamma-\gamma_i)$ *are nonzero numbers defined by* $A_2(a,b)=2(a-b,a+b)(b,\beta)^{-2}$ *and* γ, γ_1, γ_2, γ_3 *are defined in Theorem 1.*

Proof. Replacing a by $2a$, and arguing as in the proof of (32), we get

$$I_2(a,b)=2\operatorname{Re}\left(z^2(-a)\left(\sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}}\frac{\langle 2a-c,c\rangle}{(\langle c,\beta\rangle)^2}z(2a-c)z(c)\right)\right). \tag{36}$$

In (36) replacing c by $a+c$ and taking into account that $\langle a,\beta\rangle=0$, we obtain the invariant

$$I_2(a,b)=2\operatorname{Re}\left(z^2(-a)\left(\sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}}\frac{\langle a+c,a-c\rangle}{(\langle c,\beta\rangle)^2}z(a+c)z(a-c)\right)\right). \tag{37}$$

Now using this, we obtain the invariants (34) and (35) as follows. First let us consider (34). Let $a=\gamma_i,\ b=\gamma_j$. Then

$$(P(a,b)\cap Q)\backslash a\mathbb{R}=\{\pm\gamma_j,\ \pm(\gamma_i-\gamma_j),\ \pm(\gamma_i+\gamma_j)\}.$$

One the other hand, if $c=\pm(\gamma_i-\gamma_j)$, or $c=\pm(\gamma_i+\gamma_j)$, then at least one of the vectors $a-c$ and $a+c$ does not belong to Q. Therefore, the summation in (37) for this case is taken over $c\in\{\pm\gamma_j\}$ and hence (34) holds. By the third inequality in (6) we have $A_2(\gamma_i,\gamma_j)\neq 0$.

Now let us consider (35). Let $a=\gamma_i$ and $b=\gamma-\gamma_i$. Then

$$(P(a,b)\cap Q)\backslash a\mathbb{R}=\{\pm\gamma,\ \pm(\gamma-\gamma_i),\ \pm(\gamma-2\gamma_i)\}.$$

If $c=\gamma$, then $c+a=\gamma+\gamma_i\notin Q$. If $c=-\gamma$, then $c-a=-\gamma-\gamma_i\notin Q$. If $c=\gamma-2\gamma_i$, then $c-a=\gamma-3\gamma_i\notin Q$. If $c=-(\gamma-2\gamma_i)$, then $c+a=-\gamma+3\gamma_i\notin Q$. Therefore, the summation in the formula (37) for this case is taken over $c\in\{\pm(\gamma-\gamma_i)\}$ and hence (35) holds. Since $\gamma=\gamma_i+\gamma_j+\gamma_k$, it follows from the last inequality in (6) that $A_2(\gamma_i,\gamma-\gamma_i)\neq 0$. ■

3 Finding the potential from the invariants

In this section we give an algorithm for finding the all Fourier coefficients $z(a)$ of the potential (3) from the invariants (25)-(28), (34) and (35). First, let us introduce some notations. The number of elements of the set

$$\{n\gamma_1+m\gamma_2+s\gamma_3:|n|\leq 1,\ |m|\leq 1,\ |s|\leq 1\}$$

is 27, since the numbers n,m,s take 3 values $-1,0,1$ independently. The set $Q(1,1,1)$ (see (5)) is obtained from this set by eliminating the element $(0,0,0)$, and hence consist of 26 elements. Moreover, if $\gamma\in Q(1,1,1)$, then $-\gamma\in Q(1,1,1)$ and $\gamma\neq-\gamma$. Hence the elements of $Q(1,1,1)$ can be denoted by $\gamma_1,\gamma_2,...,\gamma_{13}$ and $-\gamma_1,-\gamma_2,...,-\gamma_{13}$. Let us denote the elements $\gamma_1,\gamma_2,...,\gamma_7$ as following: $\gamma_1,\gamma_2,\gamma_3$ be a basis of Γ satisfying (6) and

$$\gamma_4=\gamma_2+\gamma_3,\ \gamma_5=\gamma_1+\gamma_3,\ \gamma_6=\gamma_1+\gamma_2,\ \gamma_7=\gamma_1+\gamma_2+\gamma_3. \tag{38}$$

Introduce the notations

$$z(\gamma_j)=a_j+ib_j=r_je^{i\alpha_j}, \tag{39}$$

where $a_j\in\mathbb{R}$, $b_j\in\mathbb{R}$, $r_j=\mid z(\gamma_j)\mid\in(0,\infty)$, and $\alpha_j=\alpha(\gamma_j)=\arg(z(\gamma_j))\in[0,2\pi)$ for $i=1,2,...,13$. Since the modulus r_j of the Fourier coefficients $z(\gamma_j)$ are known due to (20), we need to know the values of the arguments α_j of $z(\gamma_j)$. For this we use the following conditions on the arguments $\alpha_1, \alpha_2, .., \alpha_7:$

$$\alpha_7 - \alpha_1 - \alpha_2 - \alpha_3 \neq \pi k,\ \alpha_7 - \alpha_{s+3} - \alpha_s \neq \pi k,\ \alpha_{m+3} - \alpha_{j+3} + \alpha_m - \alpha_j \neq \pi k,$$
$$\alpha_4 - \alpha_2 - \alpha_3 \neq \frac{\pi}{2}k,\ \alpha_5 - \alpha_1 - \alpha_3 \neq \frac{\pi}{2}k,\ \alpha_6 - \alpha_1 - \alpha_2 \neq \frac{\pi}{2}k, \tag{40}$$
$$\alpha_4 + \alpha_5 - \alpha_1 - \alpha_2 - 2\alpha_3 \neq \pi k,\ \alpha_4 + \alpha_6 - \alpha_1 - \alpha_3 - 2\alpha_2 \neq \pi k,$$
$$\alpha_5 + \alpha_6 - \alpha_2 - \alpha_3 - 2\alpha_1 \neq \pi k,$$

where $s = 1, 2, 3;$ $k \in \mathbb{Z}$ and m, j are integers satisfying $1 \leq m < j \leq 3$. In this section we give an algorithm for the unique (modulo (2)) determination of the potentials q of the form (3) satisfying (40) from the invariants (15)-(17). In the following remark we consider geometrically the set of all potentials of the form (3) satisfying (40).

Remark 1 *Since $z(\gamma) = \overline{z(-\gamma)}$, there exists one to one correspondence between the trigonometric polynomials of the form (3) and the vectors $(r_1, \alpha_1, r_2, \alpha_2, ..., r_{13}, \alpha_{13})$ of the subset*

$$S =: (0, \infty)^{13} \otimes [0, 2\pi)^{13}$$

of the space $\mathbb{R}^{26}$. We use conditions (40) as restrictions on the potential (3) and hence on the set S. Denote by $S^{'}$ the subset of S corresponding to the set of the potential (3) satisfying conditions (40). The conditions (40) means that we eliminate from the subset

$$D =: \{(\alpha_1, \alpha_2, ..., \alpha_7) : \alpha_1 \in [0, 2\pi), \alpha_1 \in [0, 2\pi),\ \alpha_2 \in [0, 2\pi), ...,\ \alpha_7 \in [0, 2\pi)\}$$

of $\mathbb{R}^7$ the following six-dimensional hyperplanes

$$\{\alpha_7 - \alpha_1 - \alpha_2 - \alpha_3 = \pi k\},\ \{\alpha_7 - \alpha_{s+3} - \alpha_s = \pi k\},\ \{\alpha_{m+3} - \alpha_{j+3} + \alpha_m - \alpha_j = \pi k\},$$
$$\{\alpha_4 - \alpha_2 - \alpha_3 = \frac{\pi}{2}k\},\ \{\alpha_5 - \alpha_1 - \alpha_3 = \frac{\pi}{2}k\},\ \{\alpha_6 - \alpha_1 - \alpha_2 = \frac{\pi}{2}k\},$$
$$\{\alpha_4 + \alpha_5 - \alpha_1 - \alpha_2 - 2\alpha_3 = \pi k\},\ \{\alpha_4 + \alpha_6 - \alpha_1 - \alpha_3 - 2\alpha_2 = \pi k\},$$
$$\{\alpha_5 + \alpha_6 - \alpha_2 - \alpha_3 - 2\alpha_1 = \pi k\}$$

of $\mathbb{R}^7 = \{(\alpha_1, \alpha_2, ..., \alpha_7)\}$, where $s = 1, 2, 3;$ $k \in \mathbb{Z}$ and m, j are integers satisfying $1 \leq m < j \leq 3$. In this notation we have

$$S = (0, \infty)^{13} \otimes [0, 2\pi)^6 \otimes D,\ S^{'} = (0, \infty)^{13} \otimes [0, 2\pi)^6 \otimes D^{'},$$

where $D^{'}$ is obtained from D by eliminating the above six-dimensional hyperplanes. It is clear that the 26 dimensional measure of the set $S \backslash S^{'}$ is zero. Since the main result (Theorem 4) of this section is concerned to the potentials corresponding to the set $S^{'}$, we investigate the almost all potentials of the form (3).

Since the operators $L(q(x - \tau))$ for $\tau \in F$ have the same band functions, we may fix τ, that is, take one of the functions $q(x - \tau)$, which determines three of the arguments.

Theorem 3 *There exists a unique value of $\tau \in F$ such that the following conditions hold*

$$\alpha(\tau, \gamma_1) = \alpha(\tau, \gamma_2) = \alpha(\tau, \gamma_3) = 0, \tag{41}$$

where $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of the lattice Γ and $\alpha(\tau, \gamma) = \arg(q(x - \tau), e^{i\langle\gamma, x\rangle})$.

Proof. Let $\omega_1, \omega_2, \omega_3$ be a basis of Ω satisfying

$$\langle\gamma_i, \omega_j\rangle = 2\pi\delta_{i,j} \tag{42}$$

and $F = \{c_1\omega_1 + c_2\omega_2 + c_3\omega_3 : c_k \in [0,1), k = 1,2,3\}$ be a fundamental domain $\mathbb{R}^3/\Omega$ of Ω. If $\tau \in F$, then we have $\tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3$. Therefore, using the notations of (3), (4) and (41) one can readily see that

$$\alpha(\tau,\gamma) = \arg(q(x-\tau), e^{i\langle\gamma,x-\tau\rangle}e^{i\langle\gamma,\tau\rangle}) = \alpha(\gamma) - \langle\gamma,\tau\rangle. \tag{43}$$

This with (42) yields $\alpha(\tau,\gamma_k) = \alpha(\gamma_k) - 2\pi c_k$ which means that (41) is equivalent to

$2\pi c_k = \alpha(\gamma_k)$, where $\alpha(\gamma_k) \in [0,2\pi)$, $2\pi c_k \in [0,2\pi)$ and $k = 1,2,3$. Thus, there exists a unique value of $\tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3 \in F$ satisfying (41) ■

By Theorem 3, without loss of generality, it can be assumed that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0. \tag{44}$$

Thus $z(\gamma_i) =| z(\gamma_i) |$ and by (20) $z(\gamma_i)$ for $i = 1,2,3$ are the known positive numbers:

$$z(\gamma_i) = a_i > 0, \ \forall i = 1,2,3. \tag{45}$$

Using (43) one can easily verify that the expressions in the left-hand sides of the inequalities in (40) do not depend on τ. Therefore, using the assumption (44) one can readily see that the condition (40) has the form

$$\alpha_7 \neq \pi k, \ \alpha_s \neq \frac{\pi}{2}k, \ \alpha_7 - \alpha_s \neq \pi k, \ \alpha_m \pm \alpha_j \neq \pi k, \tag{46}$$

where $k \in \mathbb{Z}$; $s = 4,5,6$; $j = 4,5,6$; $m = 4,5,6$ and $m \neq j$. Using the notation of (39) and taking into account that $r_j r_m \sin(\alpha_j \pm \alpha_m) = b_j a_m \pm b_m a_j$, $r_j r_m \neq 0$ (see (4)), we see that (46) can be written in the form

$$b_7 \neq 0, \ a_s b_s \neq 0, \ b_7 a_s - a_7 b_s \neq 0, \ b_j a_m \pm b_m a_j \neq 0, \tag{46.1}$$

where $s = 4,5,6$; $j = 4,5,6$; $m = 4,5,6$ and $m \neq j$.

The equality $(q(-x), e^{i\langle a,x\rangle}) = \overline{(q(x), e^{i\langle a,x\rangle})}$ shows that the imaginary part of the Fourier coefficients of $q(x)$ and $q(-x)$ take the opposite values. Therefore, taking into account the first inequality of (46.1), for fixing the inversion $q(x) \longrightarrow q(-x)$, in the set of potentials of the form (3) satisfying (40), we assume that

$$b_7 > 0. \tag{47}$$

Now using (44), (46.1), (47) and the invariants (25)-(28), (34), (35), we will find the Fourier coefficients $z(a)$ for all $a \in Q$.

Theorem 4 *The invariants (15)-(17) determine constructively and uniquely, modulo inversion and translation (2), all the potentials of the form (3) satisfying (40).*

Proof. To determine the potential (3), we find its Fourier coefficients step by step.

Step 1. In this step using (25), (27), (46.1) and (47), we find

$$z(\gamma_1+\gamma_2), \ z(\gamma_1+\gamma_3), \ z(\gamma_2+\gamma_3), \ z(\gamma_1+\gamma_2+\gamma_3). \tag{48}$$

Since $z(\gamma_1)$, $z(\gamma_2)$, $z(\gamma_3)$ are known positive numbers (see (45)), the invariants in (25) give the real parts of the Fourier coefficients $z(\gamma_2+\gamma_3), z(\gamma_1+\gamma_3), z(\gamma_1+\gamma_2)$. Then, using (20), we find the absolute values of the imaginary parts of these Fourier coefficients. Thus due to the notations of (38) and (39), we have

$$z(\gamma_2+\gamma_3) = a_4 + it_4 \mid b_4 \mid, \ z(\gamma_1+\gamma_3) = a_5 + it_5 \mid b_5 \mid, \ z(\gamma_1+\gamma_2) = a_6 + it_6 \mid b_6 \mid, \tag{49}$$

where a_m and $\mid b_m \mid$ for $m = 4, 5, 6$ are known real numbers and t_m is the sign of b_m, that is, is either -1 or 1. To determine t_4, t_5, t_6, we use (27). Using (45), (49) and the notations $\gamma = \gamma_1 + \gamma_2 + \gamma_3 = \gamma_7$, $z(\gamma_7) = a_7 + ib_7$ (see (38), (39)) one sees that (27) for $i = 1$ give us the value of $\operatorname{Re}(a_7 - ib_7)(a_4 + it_4 \mid b_4 \mid)a_1$. In other word, we have the equation

$$a_4a_7 + t_4 \mid b_4 \mid b_7 = c_1 \tag{50}$$

with respect to the unknowns a_7 and b_7, where c_1 is the known constant, since (27) is a given invariant. Here and in the forthcoming equations by c_k for $k = 1, 2, ...$ we denote the known constants. In the same way, from (27) for $i = 2, 3$, we obtain

$$a_5a_7 + t_5 \mid b_5 \mid b_7 = c_2, \tag{51}$$

$$a_6a_7 + t_6 \mid b_6 \mid b_7 = c_3. \tag{52}$$

By (46.1) $t_5 \mid b_5 \mid a_4 - t_4 \mid b_4 \mid a_5 \neq 0$, $t_6 \mid b_6 \mid a_4 - t_4 \mid b_4 \mid a_6 \neq 0$, $t_6 \mid b_6 \mid a_5 - t_5 \mid b_5 \mid a_6 \neq 0$. Therefore finding b_7 from the systems of equations generated by pairs $\{(50), (51)\}$, $\{(50), (52)\}$, $\{(51), (52)\}$, and taking into account (47), we get the inequalities

$$\frac{a_4c_2 - a_5c_1}{t_5 \mid b_5 \mid a_4 - t_4 \mid b_4 \mid a_5} > 0, \ \frac{a_4c_3 - a_6c_1}{t_6 \mid b_6 \mid a_4 - t_4 \mid b_4 \mid a_6} > 0, \ \frac{a_5c_3 - a_6c_2}{t_6 \mid b_6 \mid a_5 - t_5 \mid b_5 \mid a_6} > 0 \tag{53}$$

respectively. Now we prove that the relations (50)-(53) determines uniquely the unknowns a_7,b_7, t_4, t_5, t_6. Suppose to the contrary that there exists to different solutions $(a_7, b_7, t_4, t_5, t_6)$ and $(a_7^{'}, b_7^{'}, t_4^{'}, t_5^{'}, t_6^{'})$ of (50)-(53). Clearly, if 2 components of the triple $(t_4^{'}, t_5^{'}, t_6^{'})$ take the opposite values of the corresponding components of the triple (t_4, t_5, t_6) then all the inequalities in (53) do not hold simultaneously. Therefore, at least, two component of $(t_4^{'}, t_5^{'}, t_6^{'})$ must be the same with the corresponding two components of (t_4, t_5, t_6). It can be assumed, without loss of generality, that $t_4^{'} = t_4$ and $t_5^{'} = t_5$. Then it follows from the system of equation (50), (51) that $a_7^{'} = a_7$, $b_7^{'} = b_7$. Since $b_6b_7 \neq 0$ due to (46.1) it follows from (52) that $t_6^{'} = t_6$. Thus the Fourier coefficients in (48) can be determined from (50)-(53):

$$z(\gamma_2 + \gamma_3) = a_4 + ib_4, \ z(\gamma_1 + \gamma_3) = a_5 + ib_5, \ z(\gamma_1 + \gamma_2) = a_6 + ib_6, \tag{54}$$

$$z(\gamma_1 + \gamma_2 + \gamma_3) = a_7 + ib_7. \tag{55}$$

Step 2. In this step using (34) and (46.1), we find

$$z(\gamma_1 - \gamma_2), \ z(\gamma_1 - \gamma_3), \ z(\gamma_2 - \gamma_3). \tag{56}$$

Writing (34) for $i = 1$, $j = 2$ and taking into account that $z(-\gamma_i) = z(\gamma_i) = a_i$ (see (45)) and $z(\gamma_1 + \gamma_2) = a_6 + ib_6$ (see (54)), we find the value of $a_1^2 \operatorname{Re}(a_6 + ib_6)z(\gamma_1 - \gamma_2)$. In other word, we have an equation

$$a_6x - b_6y = c_4, \tag{57}$$

where $z(\gamma_1 - \gamma_2) = x + iy$. From (34) for $i = 2$, $j = 1$, in the same way, we get

$$a_6x + b_6y = c_5. \tag{58}$$

Since $a_6b_6 \neq 0$ due to (46.1), from (57) and (58), we find x and y and hence $z(\gamma_1 - \gamma_2)$. Similarly, writing (34) for $i = 1$, $j = 3$ and for $i = 3$, $j = 1$ we find $z(\gamma_1 - \gamma_3)$. Then, writing (34) for $i = 2$, $j = 3$ and for $i = 3$, $j = 2$, we find $z(\gamma_2 - \gamma_3)$.

Step 3. In this step using (28), (35), we find

$$z(\gamma_1 + \gamma_2 - \gamma_3), \ z(\gamma_1 + \gamma_3 - \gamma_2), \ z(\gamma_2 + \gamma_3 - \gamma_1). \tag{59}$$

Writing (28) and (35) for $i = 1$, and taking into account that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, we get

$$\mathrm{Re}(z(\gamma_2 + \gamma_3 - \gamma_1)z(-\gamma_2 - \gamma_3)z(\gamma_1)) = c_6, \tag{60}$$

$$\mathrm{Re}(z^2(-\gamma_1)z(\gamma_1 + \gamma_2 + \gamma_3)z(\gamma_1 - \gamma_2 - \gamma_3)) = c_7. \tag{61}$$

Let $z(\gamma_2 + \gamma_3 - \gamma_1) = x + iy$. Then $z(\gamma_1 - \gamma_2 - \gamma_3) = x - iy$. Now using (45), (54) and (55) from (60) and (61), we obtain the equations

$$a_4 x + b_4 y = c_8, \tag{62}$$

$$a_7 x + b_7 y = c_9. \tag{63}$$

Since $a_4 b_7 - b_4 a_7 \neq 0$, due to (46.1), from (62) and (63) we find x and y and hence $z(\gamma_2 + \gamma_3 - \gamma_1)$. In the same way, namely writing (28), (35) for $i = 2$ and for $i = 3$, we find $z(\gamma_1 + \gamma_3 - \gamma_2)$ and $z(\gamma_1 + \gamma_2 - \gamma_1)$. ■

4 On the Stability of the Algorithm

We determine constructively the potential from the band functions in two steps. In the first step we have determined the invariants from the band functions in the paper [10]. In the second step we found the potential from the invariants in Section 3 of this paper. In this section we consider the stability of the problems studied in both steps.

First, using the asymptotic formulas (13), (19) and (4) of the paper [10], denoted here as (13[10]), (19[10]) and (4[10]), we consider the stability of the invariants (15)-(17) with respect to the errors in the Bloch eigenvalues for the potential of the form (3). For this let us recall the formulas of [10] that will be used here. In [10] the spectral invariants are expressed by the band functions of the Schrödinger operator $L(q^\delta)$ with the directional potential $q^\delta(x)$ (see (11)), where δ is a visible element of Γ. The function q^δ depends only on one variable $s = \langle \delta, x \rangle$ and can be written as

$$q^\delta(x) = Q^\delta(\langle \delta, x \rangle), \text{ where } Q^\delta(s) = \sum_{n \in \mathbb{Z}} z(n\delta) e^{ins}, \tag{64}$$

that is, $Q^a(s)$ is obtained from the right-hand side of (11) by replacing $\langle a, x \rangle$ with s. The band functions and the Bloch functions of the operator $L(q^\delta)$ are

$$\lambda_{j,\beta}(v,\tau) = \mid \beta + \tau \mid^2 + \mu_j(v), \ \Phi_{j,\beta}(x) = e^{i\langle \beta + \tau, x \rangle} \varphi_{j,v}(s),$$

where $\beta \in \Gamma_\delta$, $\tau \in F_\delta =: H_\delta / \Gamma_\delta$, $j \in \mathbb{Z}$, $v \in [0, 1)$, $\mu_j(v)$ and $\varphi_{j,v}(s)$ are the eigenvalues and eigenfunctions of the operator $T_v(Q^\delta)$ generated by the boundary value problem:

$$- \mid \delta \mid^2 y''(s) + Q^\delta(s) y(s) = \mu y(s), \ y(2\pi) = e^{i2\pi v} y(0), \ y'(2\pi) = e^{i2\pi v} y'(0).$$

In the paper [10] we constructed a set of eigenvalue, denoted by $\Lambda_{j,\beta}(v,\tau)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v,\tau) = |\beta + \tau|^2 + \mu_j(v) + \frac{1}{4} \int_F |f_{\delta,\beta+\tau}|^2 |\varphi_{j,v}|^2 \, dx + O(\rho^{-3a+2\alpha_1} \ln \rho), \tag{13[10]}$$

where $\beta \sim \rho$, $j = O(\rho^{\alpha_1})$, $\alpha_1 = 3\alpha$, $a = 406\alpha$, $\alpha = \frac{1}{432}$, $-3a + 2\alpha_1 = -\frac{101}{36}$ and

$$f_{\delta,\beta+\tau}(x) = \sum_{\gamma : \gamma \in Q(1,1,1) \setminus \delta \mathbb{R}} \frac{\gamma}{\langle \beta + \tau, \gamma \rangle} z(\gamma) e^{i \langle \gamma, x \rangle}. \tag{65}$$

We say that $a(\rho)$ is of order $b(\rho)$ and write $a(\rho) \sim b(\rho)$ if there exist positive constants c_1 and c_2 such that $c_1 \mid b(\rho) \mid < \mid a(\rho) \mid < c_2 \mid b(\rho) \mid$ for $\rho \gg 1$. To consider the stability of the invariants (15)-(17) with respect to the errors in the band functions, we use (13[10]) and the following asymptotic decomposition of $\mu_j(v)$ and $|\varphi_{j,v}(s)|^2$:

$$\mu_j(v) = \mid j\delta \mid^2 + \frac{c_1}{j} + \frac{c_1}{j^2} + ... + \frac{c_n}{j^n} + O(\frac{1}{j^{n+1}}), \tag{AD1}$$

$$|\varphi_{j,v}(s)|^2 = A_0 + \frac{A_1(s)}{j} + \frac{A_2(s)}{j^2} + ... + \frac{A_n(s)}{j^n} + O(\frac{1}{j^{n+1}}), \tag{AD2}$$

where

$$c_1 = c_2 = 0,\ c_3 = \frac{1}{16\pi \mid \delta \mid^3} \int_0^{2\pi} \left| Q^\delta(t) \right|^2 dt \tag{66}$$

(see [7] and [1]). In [10] we proved that if $q^\delta(x)$ has the form (12), then

$$A_0 = 1,\ A_1 = 0,\ A_2 = \frac{Q^\delta(s)}{2} + a_1 \left| z(\delta) \right|^2,\ A_3 = a_2 Q^\delta(s) + a_3 \left| z(\delta) \right|^2, \tag{19[10]}$$
$$A_4 = a_4 Q^\delta(s) + a_5((z(\delta))^2 e^{i2\langle\delta,x\rangle} + (z(-\delta))^2 e^{-i2\langle\delta,x\rangle}) + a_6,$$

where $a_1, a_2, ..., a_6$ are the known constants.

Theorem 5 *Let $q(x)$ be the potential of the form (3), satisfying (40). If the band functions of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-\frac{101}{36}} \ln \rho)$, then one can determine the spectral invariants (15)-(17), constructively and uniquely, with accuracy $O(\rho^{-\frac{97}{108}} \ln \rho)$.*

Proof. First, using the asymptotic formula (13[10]), we write explicitly the asymptotic expression of the invariants

$$\mu_j(v),\ \ J(\delta, b, j, v) = \int_F \mid q_{\delta,b}(x) \varphi_{j,v}(\langle\delta, x\rangle) \mid^2 dx \tag{4[10]}$$

determined constructively in [10], where $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $j \in \mathbb{Z}$, $q_{\delta,b}(x)$ is defined in (14), $\delta \in Q(1,1,1)$ and b is a visible element of Γ_δ, in terms of the band functions with an estimate of the remainder term. Let $s_1 b_1,\ s_2 b_2, ..., s_m b_m$ be projections of the vectors of the set $Q(1,1,1)\backslash\delta\mathbb{R}$ onto the plane H_δ, where $s_i \in \mathbb{R}$ and $b_i \in \Gamma_\delta$ (see 24). If $b_i \in b_j\ \mathbb{R}$, where $i > j$, then we do not include b_i to the list of projections, that is, $b_1,\ b_2, ..., b_m$ are pairwise linearly independent. Consider the planes $P(\delta, b_k)$ for $k = 1, 2, ..., m$. It is clear that the set $Q(1,1,1)\backslash\delta\mathbb{R}$ is the union of the pairwise disjoint sets $P(\delta, b_k) \cap (Q\backslash\delta\mathbb{R})$ for $k = 1, 2, ..., m$. To find the spectral invariants (4[10]), we write $f_{\delta,\beta+\tau}(x)$ (see (65)) in the form

$$f_{\delta,\beta+\tau}(x) = \sum_{k=1}^{m} F_{\delta,b_k,\beta+\tau}(x), \tag{67}$$

where

$$F_{\delta,b_k,\beta+\tau}(x) = \sum_{\gamma:\gamma\in P(\delta,b_k)\cap(Q\backslash\delta\mathbb{R})} \frac{\gamma}{\langle\beta+\tau, \gamma\rangle} z(\gamma) e^{i\langle\gamma,x\rangle}. \tag{68}$$

Clearly, if $\gamma \in P(\delta, b_k)\backslash\delta\mathbb{R}$ and $\gamma' \in P(\delta, b_l)\backslash\delta\mathbb{R}$ for $l \neq k$, then $\gamma' + \gamma \notin \delta\mathbb{R}$. Therefore taking into account that $\varphi_{j,v}(\langle\delta, x\rangle)$ is a function of $\langle\delta, x\rangle$, we obtain

$$\int_F \langle\ F_{\delta,b_k,\beta+\tau}(x),\ F_{\delta,b_l,\beta+\tau}(x)\rangle \mid \varphi_{j,v}(\langle\delta, x\rangle) \mid^2 dx = 0,\ \forall l \neq k.$$

This with (67) implies that

$$\int_F \mid f_{\delta,\beta+\tau} \mid^2 |\varphi_{j,v}|^2 \, dx = \sum_{k=1}^{m} \int_F \mid F_{\delta,b_k,\beta+\tau} \mid^2 |\varphi_{j,v}|^2 \, dx \tag{69}$$

In [10] (see (58) of [10]) we proved that for each $b_0 \in \Gamma_\delta$ there exists $\beta_0 + \tau$ such that

$$\mid \beta_0 + \tau \mid \sim \rho, \ \frac{1}{3}\rho^a <\mid \langle \beta_0 + \tau, b_0 \rangle \mid < 3\rho^a,$$

and $\Lambda_{j,\beta_0}(v,\tau)$ satisfies (13[10]). Since $b_k \in \Gamma_\delta$, there exist $\beta_k + \tau$ such that

$$\frac{1}{3}\rho^a <\mid \langle \beta_k + \tau, b_k \rangle \mid < 3\rho^a \tag{70}$$

and $\Lambda_{j,\beta_0}(v,\tau)$ satisfies (13[10]). From (70) we see that $\cos\theta_{k,k} = O(\rho^{a-1}) = o(1)$, where $\theta_{s,k}$ is the angle between the vectors $\beta_s + \tau$ and b_k. Therefore $\cos\theta_{s,k} \sim 1$ for $s \neq k$ and hence

$$\langle \beta_s + \tau, b_k \rangle \sim \rho \tag{71}$$

for all $s \neq k$. If $b_0 \notin b_1\mathbb{R} \cup b_2\mathbb{R} \cup ... \cup b_m\mathbb{R}$, then (71) holds for $k = 0$ and $s = 1, 2, ..., m$.

Now substituting the orthogonal decomposition $|\delta|^{-2}\langle\gamma,\delta\rangle\delta + |b_k|^{-2}\langle\gamma, b_k\rangle b_k$ of γ for $\gamma \in P(\delta, b_k) \cap (Q\backslash\delta\mathbb{R})$ into the denominator of the fraction in (68), and taking into account that $\beta + \tau \in H_\delta$, $\langle \beta + \tau, \delta \rangle = 0$, we obtain

$$F_{\delta,b_k,\beta+\tau}(x) = \frac{|b_k|^2}{\langle \beta + \tau, b_k \rangle} q_{\delta,b_k}(x),$$

where $q_{\delta,b_k}(x)$ is defined in (14). This with (4[10]) implies that

$$\int_F \mid F_{\delta,b_k,\beta+\tau} \mid^2 |\varphi_{j,v}|^2 \, dx = \frac{|b_k|^4}{(\langle \beta + \tau, b_k \rangle)^2} J(\delta, b_k, j, v). \tag{72}$$

Substituting (69) and (72) in (13[10]) and then instead of β writing β_s for $s = 0, 1, ..., m$, we get the system of $m + 1$ equations

$$\mu_j(v) + \sum_{k=1}^{m} \frac{|b_k|^4}{4(\langle \beta_s + \tau, b_k \rangle)^2} J(\delta, b_k, j, v) = \Lambda_{j,\beta_s}(v,\tau) + |\beta_s + \tau|^2 + O(\rho^{-3a+2\alpha_1} \ln \rho), \tag{73}$$

with respect to the unknowns $\mu_j(v), J(\delta, b_1, j, v),\ J(\delta, b_2, j, v), ..., J(\delta, b_m, j, v)$. By (70) and (71) the coefficient matrix of (73) is $(a_{i,j})$, where $a_{i1} = 1$ for $i = 1, 2, ..., m + 1$ and

$$a_{k,k} \sim \rho^{-2a},\ a_{s,k} \sim \rho^{-2},\ \forall k > 1,\ \forall s \neq k. \tag{74}$$

Expanding the determinant Δ of the matrix $(a_{i,j})$, one can readily see that the highest order term of this expansion is the product of the diagonal elements of the matrix $(a_{i,j})$ which is of order ρ^{-2ma} and the other terms of this expansions are $O(\rho^{-2m})$. Therefore, we have

$$\Delta \sim \rho^{-2ma} \tag{75}$$

Now we are going to use the fact that the right-hand side of (73) is determined with error $O(\rho^{-3a+2\alpha_1} \ln \rho)$, if the band functions of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-3a+2\alpha_1} \ln \rho)$. Let Δ_k, $\Delta_{k,0}$ and $\Delta_{k,1}$ be determinant obtained from Δ by replacing s-th elements of the k-th column by

$$\Lambda_{j,\beta_s}(v,\tau)+\mid \beta_s+\tau\mid^2+O(\rho^{-3a+2\alpha_1}\ln\rho),\ \Lambda_{j,\beta_s}(v,\tau)+\mid \beta_s+\tau\mid^2)$$

and $O(\rho^{-3a+2\alpha_1}\ln\rho)$ respectively. One can easily see that

$$\Delta_1-\Delta_{1,0}=\Delta_{1,1}=O(\rho^{-2ma-3a+2\alpha_1}\ln\rho),\ \Delta_k-\Delta_{k,0}=\Delta_{k,1}=O(\rho^{-2ma-a+2\alpha_1}\ln\rho) \quad (76)$$

for $k>1$. Therefore, solving the system (73) by the Cramer's rule and using (75), (76), we find $\mu_j(v)$ and $J(\delta,b_k,j,v)$ with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$ and $O(\rho^{-a+2\alpha_1}\ln\rho)$ respectively.

Now using (AD1) for $j\sim\rho^{\alpha_1}$, where n is chosen so that $j^{n+1}>\rho^{3a}$, and taking into account that $\mu_j(v)$ is determined with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$, we consider the invariant (15). In (AD1) replacing j by kj, for $k=1,2,...,n$, we get the system of n equations

$$\frac{c_1}{jk}+\frac{c_2}{(jk)^2}+...+\frac{c_n}{(jk)^n}=\mu_{jk}(v)+\mid jk\delta\mid^2+O(\frac{1}{j^{n+1}}), \quad (77)$$

with respect to the unknowns $c_1,c_2,...,c_n$. The coefficient matrix of this system is $(a_{i,k})$, where $a_{i,k}=\frac{c_k}{(ji)^k}$ for $i,k=1,2,...,n$. Therefore the determinant of $(a_{i,k})$ is

$$\frac{c_1}{j}\frac{c_2}{j^2}...\frac{c_n}{j^n}\det(v_{i,k}),$$

where $v_{i,k}=v_i^k, v_i=\frac{1}{i}$, that is, $(v_{i,k})$ is the Vandermonde matrix and $\det(v_{i,k})\sim 1$. Now solving the system (77) by the Cramer's rule and using the arguments used for the solving of (73), we find c_3 with an accuracy $O(\rho^{-3a+5\alpha_1}\ln\rho)$, since the elements of the third column is of order $\rho^{3\alpha_1}$ and the right-hand side of (77) is determined with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$. Thus formula (66) gives the invariant (15) with error $O(\rho^{-3a+5\alpha_1}\ln\rho)$.

To consider the invariant (16) and (17), we use (AD2), where $j\sim\rho^{\alpha_1}$ and n can be chosen so that $j^{n+1}>\rho^a$. In (AD2) replacing j by kj, for $k=1,2,...,n+1$, and using it in $J(\delta,b_s,j,v)$ (see (4[10])), we get the system of $n+1$ equations

$$J_0(\delta,b_s)+\frac{J_1(\delta,b_s)}{jk}+\frac{J_2(\delta,b_s)}{(jk)^2}+...+\frac{J_n(\delta,b_s)}{(jk)^n}=J(\delta,b_s,j,v), \quad (78)$$

with respect to the unknowns $J_0(\delta,b_s),J_1(\delta,b_s),...,J_n(\delta,b_s)$, where

$$J_k(\delta,b_s)=\int_F|q_{\delta,b_s}(x)|^2A_k(\langle\delta,x\rangle)dx.$$

In the above we proved that the write-hand side of (78) is determined with error $O(\rho^{-a+2\alpha_1}\ln\rho)$. Therefore, instead of (77) using (78) and repeating the arguments used in the finding of c_3, we find $J_0(\delta,b_s),J_1(\delta,b_s),...,J_4(\delta,b_s)$ with accuracy $O(\rho^{-a+6\alpha_1}\ln\rho)$. Then using (19 [10]), we determine the invariants (16) and (17) with the accuracy $O(\rho^{-a+6\alpha_1}\ln\rho)$, where $a-6\alpha_1=\frac{97}{108}$ ■

Now considering the proof of Theorem 4, we will show that if the invariants are given with error ε, where $\varepsilon\ll 1$, then the Fourier coefficients can be determined with error ε. For this we use the following simplest lemma.

Lemma 1 *Let $x(\varepsilon)$ and $y(\varepsilon)$ be the solution of the system of the equations*

$$(a+\varepsilon)x+(b+\varepsilon)y=e+\varepsilon,\ (c+\varepsilon)x+(d+\varepsilon)y=f+\varepsilon.$$

If $ad-cb\neq 0$, then $x(\varepsilon)=x(0)+O(\varepsilon)$ and $y(\varepsilon)=y(0)+O(\varepsilon)$.

Proof. Solving the system of equations by Cramer's rule we get

$$x(\varepsilon)=\frac{(e+\varepsilon)(d+\varepsilon)-(b+\varepsilon)(f+\varepsilon)}{(a+\varepsilon)(d+\varepsilon)-(b+\varepsilon)(c+\varepsilon)}.$$

Since $(a+\varepsilon)(d+\varepsilon)-(b+\varepsilon)(c+\varepsilon)=ad-bc+O(\varepsilon)$, $ad-bc\neq 0$ and $(e+\varepsilon)(d+\varepsilon)-(b+\varepsilon)(f+\varepsilon)=ed-bf+O(\varepsilon)$ we have $x(\varepsilon)=x(0)+O(\varepsilon)$. In the same way we get $y(\varepsilon)=y(0)+O(\varepsilon)$ ■

Theorem 6 *Let $q(x)$ be the potential of the form (3) satisfying (40). If the spectral invariants (15)-(17) are given with error ε, then the potential q can be determined constructively and uniquely, modulo (2), with error $O(\varepsilon)$, where ε is a small number.*

Proof. (a) If the invariant (15) is given with the error ε, then by using (20) and (44) we determine the Fourier coefficients $z(\gamma_1),z(\gamma_2)$ and $z(\gamma_3)$ with error $O(\varepsilon)$. It follows from the proof of (49) that if the invariants (25) and (20) are given with the error $O(\varepsilon)$, then one can determine the real parts and the absolute values of the imaginary parts of the Fourier coefficients $z(\gamma_2+\gamma_3),z(\gamma_1+\gamma_3)$ and $z(\gamma_2+\gamma_3)$ with error $O(\varepsilon)$. One can readily see from the proof of *Step 1* of Theorem 4 that the error $O(\varepsilon)$ in the a_m and $\mid b_m\mid$ for $m=4,5,6$ does not influence the determinations of the signs of t_4,t_5 and t_6. Therefore the Fourier coefficients $z(\gamma_2+\gamma_3),z(\gamma_1+\gamma_3)$ and $z(\gamma_2+\gamma_3)$ can be determined with error $O(\varepsilon)$. The Fourier coefficients in (55), (56) and (59) were determined from the systems of equations generated by pairs $\{(50),(51)\}$, $\{(60),(61)\}$ and $\{(62),(63)\}$. Moreover, by (46.1), the main determinants $a_4b_5-b_4a_5$, $2a_6b_6$ and $a_4b_7-b_4a_7$ of these systems are not zero. Thus Lemma 1 implies that if the invariants are given with the error $O(\varepsilon)$, then the Fourier coefficient in (55), (56) and (59) can be determined with error $O(\varepsilon)$. ■

The consequence of Theorem 5 and Theorem 6 is the following:

Corollary 1 *Let $q(x)$ be the potential of the form (3) satisfying (40). If the band functions of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-\frac{101}{36}}\ln\rho)$, then one can determine the potential q constructively and uniquely, modulo (2), with accuracy $O(\rho^{-\frac{97}{108}}\ln\rho)$.*

5 Uniqueness Theorems

First we consider the Hill operator $H(p)$ generated in $L_2(\mathbb{R})$ by the expression $l(q)=:-y^{''}(x)+p(x)y(x)$, when $p(x)$ is a real-valued trigonometric polynomial

$$p(x)=\sum_{s=-N}^{N}p_se^{2isx},\ p_{-s}=\overline{p_s},\ p_0=0. \tag{79}$$

Let the pair $\{\lambda_{k,1},\lambda_{k,2}\}$ denote, respectively, the k-th eigenvalues of the operator generated in $L_2[0,\pi]$ by the expression $l(q)$ and the periodic boundary conditions for k even and the anti-periodic boundary conditions for k odd. It is well-known that (see [1], Theorem 4.2.4)

$$\lambda_{0,1}=\lambda_{0,2}<\lambda_{1,1}\leq\lambda_{1,2}<\lambda_{2,1}\leq\lambda_{2,2}<\lambda_{3,1}\leq\lambda_{3,2}<...<\lambda_{n,1}\leq\lambda_{n,2}<...\ .$$

The spectrum $Spec(H(p))$ of $H(p)$ is the union of the intervals $[\lambda_{n-1,2},\lambda_{n,1}]$ for $n=1,2,....$ The interval $\gamma_n=:(\lambda_{n,1},\lambda_{n,2})$ is the n-th gaps in the spectrum of $H(p)$. Since the spectrum of the operators $H(p(x))$ and $(H(p(x+\tau))$, where $\tau\in(0,\pi)$, are the same, we may assume, without loss of generality, that $p_{-N}=p_N=\mu>0$. We use the following formula obtained in the paper [5] (see Theorem 2 in [5]) for the length $\mid \gamma_n \mid$ of the gap γ_n :

$$\mid \gamma_n \mid= \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^{\frac{n}{N}} \left|\sum_{k=0}^{N-1} A_k(n)\left(1+O\left(\frac{\ln n}{n}\right)\right)\right|, \tag{80}$$

where

$$A_k(n) = \exp\left[\frac{2ink\pi}{N} + 2n\sum_{j=1}^{N-1}\lambda_j\left(\left(\tfrac{1}{2}\mu n^{-2}\right)^{\frac{1}{N}} e^{2ik\pi/N}\right)^j\right] \tag{81}$$

and λ_j algebraically depends on the Fourier coefficients of $p(x)$.

From (81) one can readily see that

$$\mid A_k(n)\mid < \exp(an^{1-\frac{2}{N}}), \quad \mid A_k(n)\mid > \exp(-an^{1-\frac{2}{N}}),\ \forall k = 0,1,...,(N-1), \tag{82}$$

where

$$a = \sum_{j=1}^{N-1} a_j,\ a_j = \sup_k \left|\operatorname{Re}(2\lambda_j\left(\left(\tfrac{1}{2}\mu\right)^{\frac{1}{N}} e^{2ik\pi/N}\right)^j\right|. \tag{83}$$

This and (80) imply that

$$\mid \gamma_n \mid< \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^{\frac{n}{N}} 2Ne^{an^{1-\frac{2}{N}}}. \tag{84}$$

Using (82)-(84) we prove the following:

Theorem 7 *Let $\widetilde{p}(x)$ be a real-valued trigonometric polynomial of the form*

$$\widetilde{p}(x) = \sum_{s=-K}^{K} \widetilde{p}_s e^{2isx},\ \widetilde{p}_{-s} = \overline{\widetilde{p}_s},\ \widetilde{p}_{-K} = \widetilde{p}_K = \nu > 0.$$

If $Spec(H(p)) = Spec(H(\widetilde{p}))$, then $K = N$, where $p(x)$ is defined in (64).

Proof. Suppose $K \neq N$. Without less of generality, it can be assumed that $K < N$. We consider the following two cases:

Case 1: Assume that $\lambda_j = 0$ for all values of j. Then by (80) for $n = lN$ and for $l \gg 1$ we have $A_k(n) = 1$ for all k. Therefore, by (80), we have

$$\mid \gamma_n \mid= \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^{l} N\left(1+O\left(\frac{\ln n}{n}\right)\right), \forall n = lN. \tag{85}$$

Applying (84) for the length $\mid \delta_n \mid$ of the n th gap δ_n in the $Spec(H(\widetilde{p}))$, that is, replacing N and μ by K and ν respectively and arguing as in the proof of (84), we see that there exist a positive number b such that

$$\mid \delta_n \mid< \frac{4n}{\nu}\left(\frac{\nu e^2}{8n^2}\right)^{\frac{n}{K}} 2Ke^{bn^{1-\frac{2}{K}}}. \tag{86}$$

Since the fastest decreasing multiplicands of (85) and (86) are n^{-2l} and $n^{-\frac{2n}{K}}$ respectively and $K < N$, it follows from (85) and (86) for $n = lN$ that $\mid \gamma_{lN} \mid > \mid \delta_{lN} \mid$ for $l \gg 1$, which contradicts to the equality $Spec(H(q)) = Spec(H(\widetilde{p}))$.

Case 2: Assume that $\lambda_j \neq 0$ for some values of j. Let us prove that the equalities

$$\mid \gamma_{lN} \mid=\mid \delta_{lN} \mid, \mid \gamma_{lN+1} \mid=\mid \delta_{lN+1} \mid, ..., \mid \gamma_{lN+N-1} \mid=\mid \delta_{lN+N-1} \mid \tag{87}$$

for $l \gg 1$ can not be satisfied simultaneously. Suppose to the contrary that all equalities in (87) hold. Using (80), (86) and taking into account that

$$\left(\frac{\nu e^2}{8n^2}\right)^{\frac{lN+m}{K}}\left(\frac{\mu e^2}{8n^2}\right)^{-\frac{lN+m}{N}} e^{bn^{1-\frac{2}{K}}} = O(n^{-\alpha n})$$

for $0<\alpha<\frac{lN+m}{K}-\frac{lN+m}{N}$, from (87) we obtain

$$\sum_{k=0}^{N-1} A_k(lN+m)\left(1+O\left(\frac{\ln l}{l}\right)\right) = O(l^{-\alpha l}),\ \forall m=0,1,...(N-1). \tag{88}$$

Let us consider $A_k(lN+m)$ in detail. It can be written in the form

$$A_k(lN+m) = \exp\left(\frac{2imk\pi}{N}\right)e^{c_k(lN+m)},\ c_k(lN+m)=\sum_{j=1}^{N-1} M_j(k)(lN+m)^{1-\frac{2j}{N}}, \tag{89}$$

where $M_j(k)$ is a complex number. Using the mean value theorem, we get

$$c_k(lN+m)-c_k(lN) = m\sum_{j=1}^{N-1} M_j(k)(lN+\theta(k))^{-\frac{2j}{N}} = O(l^{-\frac{2}{N}}), \tag{90}$$

where $\theta(k)\in[0,m]$ for all k. Now using (89), (90) and taking into account that $e^z = 1+O(z)$ as $z\to 0$, we obtain

$$A_k(lN+m) = \exp\left(\frac{2imk\pi}{N}\right)A_k(lN)(1+O(l^{-\frac{2}{N}})). \tag{91}$$

Therefore (88) has the form

$$\sum_{k=0}^{N-1}\exp\left(\frac{2imk\pi}{N}\right)A_k(lN)\left(1+o(1)\right) = O(l^{-\alpha l}),\ m=0,1,...(N-1). \tag{92}$$

Consider (92) as a system of equations with respect to the unknowns $A_0(lN),\ A_1(lN),...,\ A_{N-1}(lN)$. Using the well-known formula for the determinant of the Vandermonde matrix $(v_{m,k})$, where $v_{m,k}=v_m^k, v_m=\exp(\frac{2im\pi}{N})$, we see that the main determinant of this system is

$$(1+o(1))\det\left(e^{\frac{2imk\pi}{N}}\right)_{k,m=0}^{N-1} = (1+o(1))\prod_{0\le m<k\le(N-1)}(e^{\frac{2ik\pi}{N}}-e^{\frac{2im\pi}{N}}).$$

Thus solving (92) by the Cramer's rule we obtain $A_k(lN)=O(l^{-\alpha l})$, for $k=0,1,...(N-1)$ which contradicts the second inequality in (82). The theorem is proved. ■

Now using this theorem we prove a uniqueness theorem for the three-dimensional Schrödinger operator. For this, first, we prove the following lemma.

Lemma 2 *Let $\widetilde{q}(x)$ be infinitely differentiable periodic potential of the form*

$$\widetilde{q}(x) = \sum_{a\in Q(1,1,1)} \widetilde{q}^a(x), \tag{93}$$

where

$$\widetilde{q}^a(x) = \sum_{n\in\mathbb{Z}} \widetilde{z}(na)e^{in\langle a,x\rangle},\ \widetilde{z}(0)=0 \tag{94}$$

and $\widetilde{z}(na) =: (\widetilde{q}(x), e^{in\langle a,x\rangle})$ *is the Fourier coefficients of* $\widetilde{q}$. *If the equalities*

$$\widetilde{z}(n\gamma_i) = 0,\ \widetilde{z}(n\gamma_j) = 0,\ \forall n \in \mathbb{Z}\backslash\{-1, 1\} \tag{95}$$

hold, then

$$\widetilde{I}_1(\gamma_i + \gamma_j,\ \gamma_i) = A_1(\gamma_i + \gamma_j,\ \gamma_i)\,\mathrm{Re}(\widetilde{z}(-\gamma_i - \gamma_j)\widetilde{z}(\gamma_i)\widetilde{z}(\gamma_j)), \tag{$\widetilde{25}$}$$

$$\widetilde{I}_1(\gamma_i - \gamma_j,\ \gamma_i) = A_1(\gamma_i - \gamma_j,\ \gamma_i)\,\mathrm{Re}(\widetilde{z}(-\gamma_i + \gamma_j)\widetilde{z}(\gamma_i)\widetilde{z}(-\gamma_j)), \tag{$\widetilde{26}$}$$

$$\widetilde{I}_2(\gamma_i,\ \gamma_j) = A_2(\gamma_i, \gamma_j)\,\mathrm{Re}(\widetilde{z}(-\gamma_i))^2\widetilde{z}(\gamma_i + \gamma_j)\widetilde{z}(\gamma_i - \gamma_j)) \tag{$\widetilde{34}$}$$

for $i \neq j$, *where* $A_1(a, b)$ *and* $A_2(a, b)$ *are defined in Theorem 1 and Theorem 2 respectively,* $\widetilde{I}_1(a, b)$ *and* $\widetilde{I}_2(a, b)$ *are the invariants (16) and (17) for the operator* $L(\widetilde{q})$.

Proof. By definition of $\widetilde{I}_1(\gamma_i + \gamma_j,\ \gamma_i)$ (see (16) and (14)) we have

$$\widetilde{I}_1(\gamma_i + \gamma_j,\ \gamma_i) = \int_F \left|\widetilde{q}_{\gamma_i+\gamma_j,\beta}(x)\right|^2 (\widetilde{q})^{\gamma_i+\gamma_j}(x)dx, \tag{96}$$

where β is defined by (24),

$$\widetilde{q}_{\gamma_i+\gamma_j,\beta}(x) = \sum_{c\in D} \frac{c}{\langle \beta, c\rangle}\widetilde{z}(c)e^{i\langle c,x\rangle}, \tag{97}$$

and $D = \{c \in (P(\gamma_i, \gamma_j) \cap \Gamma)\backslash(\gamma_i + \gamma_j)\mathbb{R} : \widetilde{z}(c) \neq 0\}$. It follows from (93) that if $c \in D$, then $c = ka$, where k is an integer, and a belongs to the set $P(\gamma_i, \gamma_j) \cap Q)\backslash(\gamma_i + \gamma_j)\mathbb{R}$. Since this set is $\{\gamma_i, \gamma_j, -\gamma_i, -\gamma_j, \gamma_i - \gamma_j, -(\gamma_i - \gamma_j)\}$ and (95) holds, we have

$$D = \{\gamma_i, \gamma_j, -\gamma_i, -\gamma_j\} \cup \{k(\gamma_i - \gamma_j) : k \in \mathbb{Z}\}. \tag{98}$$

Therefore, repeating the proof of (32), we see that

$$\widetilde{I}_1(\gamma_i + \gamma_j,\ \gamma_i) = 2\,\mathrm{Re}\left(\sum_{n=1}^{\infty} \widetilde{z}(-n(\gamma_i + \gamma_j)) \sum_{c\in D} \frac{\langle n(\gamma_i + \gamma_j) - c, c\rangle}{(\langle c, \beta\rangle)^2}\widetilde{z}(n(\gamma_i + \gamma_j) - c)\widetilde{z}(c)\right). \tag{99}$$

It follows from (98) that if $n > 1$ and $c \in D$, then $n(\gamma_i+\gamma_j)-c \notin D$ and $\widetilde{z}(n(\gamma_i+\gamma_j)-c) = 0$. Hence, from (99) we obtain

$$\widetilde{I}_1(\gamma_i + \gamma_j,\ \gamma_i) = 2\,\mathrm{Re}\left(\widetilde{z}(-(\gamma_i + \gamma_j)) \sum_{c\in D} \frac{\langle(\gamma_i + \gamma_j) - c, c\rangle}{(\langle c, \beta\rangle)^2}\widetilde{z}((\gamma_i + \gamma_j) - c)\widetilde{z}(c)\right). \tag{100}$$

Using this instead of (32) and repeating the proof of (25), we get $(\widetilde{25})$. In $(\widetilde{25})$ replacing γ_j by $-\gamma_j$, we get $(\widetilde{26})$.

Now let us prove $(\widetilde{34})$. It follows from (95) that

$$(\widetilde{q})^{\gamma_i}(x) = \widetilde{z}(\gamma_i)e^{i\langle\gamma_i,x\rangle} + \widetilde{z}(-\gamma_i)e^{-i\langle\gamma_i,x\rangle}).$$

Therefore $\widetilde{I}_2(\gamma_i, \gamma_j)$ has the form

$$\widetilde{I}_2(\gamma_i, \gamma_j) = \int_F |\widetilde{q}_{\gamma_i,\beta}(x)|^2 (((\widetilde{z}(\gamma_i))^2 e^{i2\langle\gamma_i,x\rangle} + (\widetilde{z}(-\gamma_i))^2 e^{-i2\langle\gamma_i,x\rangle})dx \tag{101}$$

(see (17)), where β is defined by (24),

$$\widetilde{q}_{\gamma_i,\beta}(x)=\sum_{c\in E}\frac{c}{\langle\beta,c\rangle}\widetilde{z}(c)e^{i\langle c,x\rangle},\tag{102}$$

$E=\{c\in(P(\gamma_i,\gamma_j)\cap\Gamma)\backslash\gamma_i\mathbb{R}:\widetilde{z}(c)\neq 0\}$. Arguing as in the proof of (98), (99), we see that

$$E=\{\gamma_j,-\gamma_j\}\cup\{k(\gamma_i-\gamma_j):k\in\mathbb{Z}\}\cup\{n(\gamma_i+\gamma_j):n\in\mathbb{Z}\}.\tag{103}$$

$$\widetilde{I}_2(\gamma_i,\gamma_j)=2\operatorname{Re}\left(\widetilde{z}^2(-\gamma_i)\sum_{c\in E}\frac{\langle\gamma_i+c,\gamma_i-c\rangle}{(\langle c,\beta\rangle)^2}\widetilde{z}(\gamma_i+c)\widetilde{z}(\gamma_i-c)\right).\tag{104}$$

If $c=k(\gamma_i-\gamma_j)$, where $k\neq 0$, or $c=n(\gamma_i+\gamma_j)$, where $n\neq 0$, then at least one of the vectors γ_i-c and γ_i+c does not have the form $c=sa$, where $s\in\mathbb{Z}$, $a\in P(\gamma_i,\gamma_j)\cap Q)\backslash\gamma_i\mathbb{R}$, and hence by (93) we have $\widetilde{z}(\gamma_i+c)\widetilde{z}(\gamma_i-c)=0$. Therefore, the summation in (104) is taken over $c\in\{\pm\gamma_j\}$ and $\widetilde{(34)}$ holds. ■

Now we prove a uniqueness theorem for the periodic, with respect to the lattice Ω, potentials $q(x)$ of $C^1(\mathbb{R}^3)$ subject to some constraints only on the directional potentials (see (10), (11)) $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$ and $q^{\gamma_3}(x)$, where $\{\gamma_1,\gamma_2,\gamma_3\}$ is a basis of Γ satisfying (6). Note that the directional potential $q^a(x)$ is a function $Q^a(s)$ of one variable $s=:\langle x,a\rangle\in\mathbb{R}$, where the function $Q^a(s)$ is obtained from the right-hand side of (11) by replacing $\langle x,a\rangle$ with s, that is, $Q^a(\langle x,a\rangle)=q^a(x)$ (see (64)). Let M be the set of all periodic,with period 2π, functions $f\in C^1(\mathbb{R})$ such that $spec(H(f))=spec(H(\mu\cos s))$ for some positive μ. Denote by W the set of all periodic, with respect to the lattice Ω, functions $q(x)$ of $C^1(\mathbb{R}^3)$ whose directional potentials $q^{\gamma_k}(x)$ for $k=1,2,3$ satisfy the conditions

$$Q^{\gamma_k}\in(C^1(\mathbb{R})\backslash M)\cup P,\ \forall k=1,2,3,\tag{105}$$

where P is the set of all trigonometric polynomial. Thus we put condition only on the directional potentials $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$ and $q^{\gamma_3}(x)$. The all other directional potentials, that is, $q^a(x)$ for all $a\in S\backslash\{\gamma_1,\gamma_2,\gamma_3\}$, where S is the set of all visible elements of Γ, are arbitrary continuously differentiable functions.

Theorem 8 *Let $q(x)$ be the potential of the form (3), satisfying (40). If $\widetilde{q}\in W$ and the band functions of the operators $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ is equal to q modulo (2).*

Proof. Let $\widetilde{q}$ be a function of W whose band functions coincides with the band functions of q. By Theorem 6.1 of [2] the band functions of $L(\widetilde{q}^a)$ coincides with the band functions of $L(q^a)$. It implies that the spectrum of $H(\widetilde{Q}^a)$ coincides with the spectrum of $H(Q^a)$, where $\widetilde{Q}^a(\langle x,a\rangle)=\widetilde{q}^a(x)$. Since the length of the n-th gap in the spectrum of $H(Q^a)$ satisfies (84), the same formula holds for the n-th gap of $H(\widetilde{Q}^a)$. It implies that $\widetilde{q}^a$ is an infinitely differentiable function for all visible elements a of Γ (see [7]). Thus $\widetilde{q}(x)$ is an infinitely differentiable function and due to [10] the operator $L(\widetilde{q})$ has the invariants (15)-(17) denoted by $\widetilde{I}(a)$, $\widetilde{I}_1(a,b)$, $\widetilde{I}_2(a,b)$. Since the band functions of $L(q)$ and $L(\widetilde{q})$ coincide, we have

$$Spec(H(\widetilde{Q}^a))=Spec(H(Q^a)),\ \widetilde{I}(a)=I(a),\widetilde{I}_1(a,b)=I_1(a,\beta),\widetilde{I}_2(a,b)=I_2(a,b)\tag{106}$$

(see Theorem 5 of [10]). We need to prove that $\widetilde{q}(x)\in\{q(sx+\tau):\tau\in F,s=\pm1\}$. For this, it is enough to show that there exist $\tau\in F,s\in\{-1,1\}$ such that $\widetilde{q}(sx-\tau)=q(x)$. The draft scheme of the proof is the followings. In Theorem 4 we proved that if $q(x)$ has the form (3), then its Fourier coefficients $z(a)$ for $a\in Q(1,1,1)$ can be defined uniquely, modulo (2), from the invariants (25)-(28), (34) and (35). Here we prove that if the band functions of the operators $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ has the form (3) and the operator $L(\widetilde{q})$ has the spectral invariants, denoted by $\widetilde{(25)}$-$\widetilde{(28)}$, $\widetilde{(34)}$, $\widetilde{(35)}$ and obtained from the formulas (25)-(28), (34), (35) respectively by replacing everywhere $z(a)$ with $\widetilde{z}(a)$. Then, using the arguments of the proof of Theorem 4 and fixing the inversion and translations (2), we prove that $\widetilde{z}(a) = z(a)$ for $a \in Q(1,1,1)$.

Since $q^a(x) = 0$ for $a \in S\backslash Q(1,1,1)$, the equality (15) and the second equality of (106) imply that $\widetilde{q}$ has the form (93). Now, to show that $\widetilde{q}(x)$ has the form (3), we prove that

$$\widetilde{z}(na) = 0, \ \forall \mid n \mid > 1, a \in Q(1,1,1). \tag{107}$$

By (45) we have $Q^{\gamma_k}(s) = a_k \cos s$ where $a_k > 0$ and $k = 1,2,3$. Therefore, by the first equality of (106), $\widetilde{Q}^{\gamma_k} \in M$. On the other hand, by the definition of W, we have $\widetilde{Q}^{\gamma_k} \in (C^1(\mathbb{R})\backslash M) \cup P$ (see (105). Thus $\widetilde{Q}^{\gamma_k} \in P$. Then, it follows from Theorem 7 that (107) holds for $a \in \{\gamma_1, \gamma_2, \gamma_3\}$. Hence the all conditions of Lemma 2 hold and we have the formulas $\widetilde{(25)}$, $\widetilde{(26)}$ and $\widetilde{(34)}$. Besides, it follows from the second equality of (106) that$\mid \widetilde{z}(\gamma_i) \mid = \mid z(\gamma_i) \mid$. By Theorem 3 there exists $\tau \in F$ such that

$$\arg(\widetilde{q}(x-\tau), e^{-i\langle \gamma_k, x\rangle}) = 0, \ \forall k = 1,2,3.$$

Without loss of generality, we denote $\widetilde{q}(x-\tau)$ by $\widetilde{q}$ and its Fourier coefficients by $\widetilde{z}(a)$. Thus we have

$$\widetilde{z}(\gamma_i) = z(\gamma_i) = a_i > 0, \ \forall i = 1,2,3. \tag{108}$$

These with (25), (26), $\widetilde{(25)}$, $\widetilde{(26)}$ and (108) imply that

$$\mathrm{Re}(\widetilde{z}(\gamma_i \pm \gamma_j)) = \mathrm{Re}(z(\gamma_i \pm \gamma_j)). \tag{109}$$

From this using the obvious equalities (see (15) and the second equality of (106))

$$\sum_{n=1}^{\infty} 2 \mid \widetilde{z}(n(\gamma_i \pm \gamma_j)) \mid^2 = \widetilde{I}(\gamma_i \pm \gamma_j) = I(\gamma_i \pm \gamma_j) = 2 \mid z(\gamma_i \pm \gamma_j) \mid^2, \tag{110}$$

we obtain

$$\mid \mathrm{Im}(\widetilde{z}(\gamma_i \pm \gamma_j)) \mid \leq \mid \mathrm{Im}(z(\gamma_i \pm \gamma_j)) \mid. \tag{111}$$

On the other hand, using (34), $\widetilde{(34)}$, (108) and (106), we obtain

$$\mathrm{Re}(\widetilde{z}(\gamma_i + \gamma_j)\widetilde{z}(\gamma_i - \gamma_j)) = \mathrm{Re}(z(\gamma_i + \gamma_j)z(\gamma_i - \gamma_j)).$$

This with (109) and (111) imply that

$$\mid \mathrm{Im}(\widetilde{z}(\gamma_i \pm \gamma_j)) \mid = \mid \mathrm{Im}(z(\gamma_i \pm \gamma_j)) \mid. \tag{112}$$

Thus by (109) and (112), we have

$$\mid \widetilde{z}(\gamma_i \pm \gamma_j) \mid = \mid z(\gamma_i \pm \gamma_j) \mid. \tag{113}$$

Therefore, from (110) we see that (107) holds for $a = \gamma_i \pm \gamma_j$. Hence we have

$$\widetilde{z}(n(\gamma_i \pm \gamma_j) = 0, \ \widetilde{z}(n\gamma_m) = 0, \ \forall n \in \mathbb{Z}\backslash\{-1,1\}, \tag{114}$$

where i, j, m are different integers satisfying $1 \leq i, j, m \leq 3$. Now instead of (95) using (114), that is, instead γ_i and γ_j in (95) taking $\gamma_i \pm \gamma_j$ and γ_m respectively, and repeating the proof of Lemma 2, we obtain that

$$\widetilde{I}_1(\gamma,\ \gamma_i) = A_1(\gamma,\ \gamma_i)\,\mathrm{Re}(\widetilde{z}(-\gamma)\widetilde{z}(\gamma-\gamma_i)\widetilde{z}(\gamma_i)), \qquad (\widetilde{27})$$

$$\widetilde{I}_1(2\gamma_i-\gamma,\ \gamma_i) = A_1(2\gamma_i-\gamma,\ \gamma_i)\,\mathrm{Re}(\widetilde{z}(\gamma-2\gamma_i)\widetilde{z}(\gamma_i-\gamma)\widetilde{z}(\gamma_i)), \qquad (\widetilde{28})$$

$$\widetilde{I}_2(\gamma_i,\ \gamma-\gamma_i) = A_2(\gamma_i,\gamma_j)\,\mathrm{Re}(\widetilde{z}(-\gamma_i))^2\widetilde{z}(\gamma)\widetilde{z}(2\gamma_i-\gamma)) \qquad (\widetilde{35})$$

for $i = 1, 2, 3$; $i \neq j$, where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$.

One can readily see that the formulas $(\widetilde{25})$-$(\widetilde{28})$, $(\widetilde{34})$, $(\widetilde{35})$ are obtained from the formulas (25)-(28), (34), (35) respectively by replacing everywhere $z(a)$ with $\widetilde{z}(a)$. Moreover, by (108), (109) and (112), we have

$$\widetilde{a_i} = a_i,\ \forall i = 1, 2, ..., 6;\ \widetilde{b_i} = \pm b_i,\ \forall i = 4, 5, 6, \qquad (115)$$

where $\widetilde{a_i} + i\widetilde{b_i} = \widetilde{z}(\gamma_i)$. As in *Step 1* in the proof of Theorem 4, using $(\widetilde{27})$ for $i = 1, 2, 3$ and taking into account (115), we obtain the equations

$$a_4\widetilde{a}_7 + \widetilde{t}_4 \mid b_4 \mid \widetilde{b}_7 = c_1 \qquad (\widetilde{50})$$

$$a_5\widetilde{a}_7 + \widetilde{t}_5 \mid b_5 \mid \widetilde{b}_7 = c_2, \qquad (\widetilde{51})$$

$$a_6\widetilde{a}_7 + \widetilde{t}_6 \mid b_6 \mid \widetilde{b}_7 = c_3, \qquad (\widetilde{52})$$

where $\widetilde{t}_m$ is the sign of $\widetilde{b}_m$, that is, is either -1 or 1 and c_1, c_2, c_3 are the known constants defined in (50), (51), (52). It follows from (46.1) that the main determinants of the systems of equations, with respect to the unknowns $\widetilde{a}_7$, $\widetilde{b}_7$, generated by pairs $\{(\widetilde{50}),\ (\widetilde{51})\}$, $\{(\widetilde{50}),\ (\widetilde{52})\}$, $\{(\widetilde{51}),\ (\widetilde{52})\}$ are not zero. Finding $\widetilde{b}_7$ from $(\widetilde{50})$, $(\widetilde{51})$ and taking into account (53), we see that $\widetilde{b}_7 \neq 0$. Therefore, for fixing the inversion $\widetilde{q}(x) \longrightarrow \widetilde{q}(-x)$, we assume that $\widetilde{b}_7 > 0$. Using this and finding $\widetilde{b}_7$ from the systems generated by pairs $\{(\widetilde{50}),\ (\widetilde{51})\}$, $\{(\widetilde{50}),\ (\widetilde{52})\}$, $\{(\widetilde{51}),\ (\widetilde{52})\}$, we get the inequalities

$$\frac{a_4c_2 - a_5c_1}{\widetilde{t}_5 \mid b_5 \mid a_4 - \widetilde{t}_4 \mid b_4 \mid a_5} > 0,\ \frac{a_4c_3 - a_6c_1}{\widetilde{t}_6 \mid b_6 \mid a_4 - \widetilde{t}_4 \mid b_4 \mid a_6} > 0, \frac{a_5c_3 - a_6c_2}{\widetilde{t}_6 \mid b_6 \mid a_5 - \widetilde{t}_5 \mid b_5 \mid a_6} > 0 \qquad (\widetilde{53})$$

One can readily see that the relations $(\widetilde{50})$-$(\widetilde{53})$ with respect to the unknowns $\widetilde{a}_7$, $\widetilde{b}_7$, $\widetilde{t}_4$, $\widetilde{t}_5$, $\widetilde{t}_6$ are obtained from (50)-(53) by replacing the unknowns a_7, b_7, t_4, t_5, t_6 with $\widetilde{a}_7$, $\widetilde{b}_7$, $\widetilde{t}_4$, $\widetilde{t}_5$, $\widetilde{t}_6$. Since we proved that (50)-(53) has a unique solution, we have:

$a_7 = \widetilde{a}_7$, $b_7 = \widetilde{b}_7$, $t_4 = \widetilde{t}_4$, $t_5 = \widetilde{t}_5$, $t_6 = \widetilde{t}_6$. This with (115) imply that

$$\widetilde{a_i} = a_i,\ \widetilde{b_i} = b_i,\ \forall i = 1, 2, ..., 7. \qquad (116)$$

In *Step 2* and *Step 3* of Theorem 4 using the invariants (28), (34), (35) we have determined the all other Fourier coefficients of q provided that a_i and b_i for $i = 1, 2, ..., 7$ are known. Since the invariants $(\widetilde{28})$, $(\widetilde{34})$, $(\widetilde{35})$ are obtained from the invariants (28), (34), (35) by replacing everywhere a_i and b_i with $\widetilde{a_i}$ and $\widetilde{b_i}$ respectively, and (116) holds, we have

$$\widetilde{z}(a) = z(a),\ \forall a \in Q(1, 1, 1). \qquad (117)$$

This with the equalities (15), (20), (106) and (94) imply that (107) holds for all $a \in Q(1, 1, 1)$. Therefore, it follow from (93), (107) and (117) that $\widetilde{q}(x) = q(x)$ ∎

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