Rare semileptonic $B_s$ decays to $\eta$ and $\eta'$ mesons in QCD

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We analyze the rare semileptonic $B_s \rightarrow (\eta, \eta')l^+l^-$, ($l = e, \mu, \tau$), and $B_s \rightarrow (\eta, \eta')\nu\bar{\nu}$ transitions probing the $s\bar{s}$ content of the $\eta$ and $\eta'$ mesons via three-point QCD sum rules. We calculate responsible form factors for these transitions in full theory. Using the obtained form factors, we also estimate the related branching fractions and longitudinal lepton polarization asymmetries. Our results are in a good consistency with the predictions of the other existing nonperturbative approaches.

I. INTRODUCTION

Among $B$ mesons, the $B_s$ has been received special attention, since experimentally it is expected that an abundant number of $B_s$ will be produced at LHCb. This will provide the possibility to study the properties of this meson and its various decay channels. The first evidence for $B_s$ production at the $\Upsilon(5S)$ peak was found by the CLEO Collaboration [1,2]. Recently, the Belle Collaboration measured the branching ratios of the $B_s \rightarrow J/\psi \phi$ transition as well as the $B_s \rightarrow J/\psi \eta$ decay via the $\eta \rightarrow \gamma \gamma$ and $\eta \rightarrow \pi^+ \pi^0 \pi^-$ channels to reconstruct the $\eta$ meson [3].

Semileptonic decays of the $B_s$ to the $\eta$ and $\eta'$, induced by the rare flavor changing neutral current transition of $b \rightarrow s l^+l^-$ and $b \rightarrow s \nu\bar{\nu}$ are a crucial framework to restrict the standard model (SM) parameters. They can provide the possibility of extracting the elements of the Cabbibo-Kobayashi-Maskawa matrix and search for origin of the $CP$ and $T$ violations. As these transitions occur at the lowest order through one-loop penguin diagrams, they are a good context to search for new physics effects beyond the SM. Looking for supersymmetric particles [4], light dark matter [5] and fourth generation of quarks is possible via these transitions. These transitions are also useful in studying the structures of the $\eta$ and $\eta'$ mesons.

In the present work, we analyze the semileptonic $B_s \rightarrow (\eta, \eta')l^+l^-/\nu\bar{\nu}$ decays considering also the $s\bar{s}$ content of the $\eta$ and $\eta'$ mesons in the framework of the three-point QCD sum rules. Here, we consider also the mixing between the $\eta$ and $\eta'$ mesons with a single mixing angle [6,7] as

\[ |\eta_q\rangle = \frac{1}{\sqrt{2}} (|\bar{u}u\rangle + |\bar{d}d\rangle), \quad |\eta_s\rangle = |s\bar{s}\rangle. \]  

(2)

The decay constants of $\bar{q}q$ and $s\bar{s}$ parts are defined in terms of the pion decay constant as [6]

\[ f_q = (1.02 \pm 0.02) f_\pi, \quad f_s = (1.34 \pm 0.06) f_\pi. \]  

(3)

We will use the mixing angle $\varphi = (41.5 \pm 0.3_{\text{stat}} \pm 0.7_{\text{syst}} \pm 0.6_{\text{th}})\,^\circ$ [11], which has recently been obtained by the KLOE Collaboration in the $QF$ basis via measuring the ratio $\frac{\Gamma(\phi\rightarrow\eta\gamma)}{\Gamma(\phi\rightarrow\eta')}$ in the $QF$ basis with the single mixing angle, the form factors of $B_s \rightarrow (\eta, \eta')$ transitions are defined in terms of the form factors $B_s \rightarrow \eta_s$ as

\[ f_i^{B_s \rightarrow \eta(\eta')} = -\sin\varphi (\cos\varphi) f_i^{B_s \rightarrow \eta_s}, \]  

(4)

and their branching fractions are also related to the branching ratio of $B_s \rightarrow \eta_s$ as follows:

\[ BR(B_s \rightarrow \eta(\eta')l^+l^-) = \sin^2\varphi (\cos^2\varphi) \]  

\[ \quad \text{BR}[B_s \rightarrow \eta_s l^+l^-]. \]  

(5)

The paper is organized as follows: sum rules for form factors responsible for considered transitions are obtained in Sec. II. Section III is devoted to the numerical analysis of the form factors, branching ratios, and longitudinal lepton polarization asymmetries as well as our discussions. In this section, we also compare the obtained results with the existing predictions of the other nonperturbative approaches.

II. QCD SUM RULES FOR TRANSITION FORM FACTORS

As we previously mentioned, to calculate the form factors responsible for the rare semileptonic $B_s \rightarrow (\eta, \eta')l^+l^-$, ($l = e, \mu, \tau$) and $B_s \rightarrow (\eta, \eta')\nu\bar{\nu}$ decays, we need to calculate the form factors of $B_s \rightarrow \eta_l l^+l^-/\nu\bar{\nu}$. To this aim, we start with the following three-point correlation function, which is constructed from the vacuum expectation value of time ordered product

\[ \langle 0|J \phi |f \rangle = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\epsilon_{\lambda\mu\nu}}{2\sqrt{2} \pi^2 f_\pi} \bar{u}(p') \gamma_{\lambda}(p') \gamma_\mu(\mathbf{p}) \gamma_\nu(\mathbf{p}) u(p), \]  

(6)

where $\mathbf{p}$ and $\mathbf{p}'$ are the four-momenta of $B_s$ and $\phi$, respectively. $f_\pi$ is the pion decay constant, $J$ is the current for $B_s \rightarrow \eta_l l^+l^-$ or $B_s \rightarrow \eta_s$, $\epsilon_{\lambda\mu\nu}$ is the totally antisymmetric tensor, and $\gamma_\mu(\mathbf{p})$ is the gamma matrix in the $QF$ basis.

\[ f_1(\omega) = \frac{G_{F}}{2 \sqrt{2} f_\pi} \frac{V_{us} V_{ub}^{*}}{m_b} \left[ \frac{1}{c_1(\omega)} - \frac{1}{c_3(\omega)} \right] + \frac{G_{F}}{2 \sqrt{2} f_\pi} V_{ud} V_{ub}^{*} \frac{1}{c_3(\omega)} \delta(\omega), \]  

(7)

where $\delta(\omega)$ is the pole term with the mass of $\phi$, $c_{1,3}(\omega)$ are the form factors, $c_{3}(\omega)$ is the form factor of the $B_s$ to $\eta_s$, and $G_{F}$ is the Fermi constant.

\[ c_{3}(\omega) = \frac{1}{m_b^2} \frac{c_{30} + c_{31} \omega^2}{1 + c_{32} \omega^2}, \]  

(8)

where $c_{30,1,2}$ are the constants of the three-point correlation function, and $c_{30} = 1$.

\[ c_{3}(\omega) = \frac{1}{m_b^2} \frac{1}{1 + c_{32} \omega^2} \]  

(9)

where $c_{3}(\omega)$ is the form factor of the $B_s$ to $\eta_s$.
\[ \Pi_{\mu}^{V,T} = i^2 \int \! d^4x d^4y e^{-ipx} e^{ip'y}(0 | T \{ J_{\mu}^{V,T}(y) J_{\mu}^{V,T}(0) J_{\mu}^{V,T}(x) \} | 0), \]  

(6)

where \( p \) and \( p' \) are the initial and final momentums, respectively, \( J_{\mu}^{V}(x) = \bar{s}(x) \gamma_5 b(x) \) and \( J_{\mu}^{T}(y) = \bar{s}(y) \gamma_\mu s(y) \), are the interpolating currents of the \( B_s \) and \( \eta_s \) states and \( J_{\mu}^{V}(0) = \bar{s}(0) \gamma_\mu b(0) \) and \( J_{\mu}^{T}(0) = \bar{s}(0) \gamma_\mu \sigma_{\mu \nu} q^\nu b(0) \) are the vector and tensor transition currents extracted from the effective Hamiltonian responsible for the \( B_s \rightarrow \eta_s l^+ l^- / \nu \bar{\nu} \) decays. At quark level, these transitions are governed by \( b \rightarrow s l^+ l^- \) and \( b \rightarrow s \nu \bar{\nu} \) via penguin and box diagrams (see Fig. 1). The correcting effective Hamiltonian is presented in terms of the Wilson coefficients, \( C_{\gamma}^{\text{eff}} \), \( C_{\gamma}^{\text{eff}} \) and \( C_{10} \) as

\[ \mathcal{H}_{\text{eff}} = \frac{G_F \alpha}{2\sqrt{2}\pi} V_{tb} V_{ts}^* \left[ C_{\gamma}^{\text{eff}} \bar{s} \gamma_\mu (1 - \gamma_5) b \gamma_\mu \ell + C_{10} \bar{s} \gamma_\mu (1 - \gamma_5) b \gamma_\mu \gamma_5 \ell - 2 C_{17}^{\text{eff}} \frac{m_b}{q^2} \bar{s} i \sigma_{\mu \nu} q^\nu (1 + \gamma_5) b \gamma_\mu \ell \right], \]

(7)

where \( G_F \) is the Fermi constant, \( \alpha \) is the fine structure constant at Z mass scale, and \( V_{ij} \) are elements of the Cabbibo- Kobayashi-Maskawa matrix. For the \( \nu \bar{\nu} \) case, only the term containing \( C_{10} \) is considered. It should be mentioned that because of the parity conservations, the axial vector and pseudotensor currents do not contribute to the pseudoscalar-pseudoscalar hadronic matrix element, i.e.,

\[ \langle P(p') | J_{\mu}^{V} \rangle = \bar{s} \gamma_\mu \gamma_5 b \ | B_s(p) \rangle = 0, \]

\[ \langle P(p') | J_{\mu}^{T} \rangle = \bar{s} i \sigma_{\mu \nu} q^\nu \gamma_5 b \ | B_s(p) \rangle = 0, \]

(8)

where \( P \) stands for the \( \eta(\eta') \) meson.

From the general aspect of the QCD sum rules, we calculate the aforementioned correlation function in two different ways. First, in the hadronic representation, it is calculated in the timelike region in terms of hadronic parameters called the phenomenological or physical side. Second, it is calculated in the spacelike region in terms of QCD degrees of freedom called the QCD or theoretical side. The sum rules for the form factors can be obtained equating the coefficient of the selected structures from these two representations of the same correlation function through the dispersion relation and applying the double Borel transformation with respect to the momentums of the initial and final states to suppress the contributions coming from the higher states and continuum.

In order to obtain the phenomenological representation of the correlation function given in Eq. (6), two complete sets of intermediate states with the same quantum numbers as the interpolating currents \( J_{\mu} \) and \( B_s \) are inserted to sufficient places. As a result of this procedure, we obtain

\[ \Pi_{\mu}^{V}(p^2, p'^2, q^2) = \frac{\langle 0 | J_{\mu}^{V}(p') \rangle \langle P(p') | J_{\mu}^{V}(p) \rangle | B_s(p) \rangle | J_{\mu}^{V}(0) \rangle | 0 \rangle}{(p'^2 - m_B^2)(p^2 - m_{B_s}^2)} + \cdots, \]

(9)

where \( \cdots \) represents the contributions coming from the higher states and continuum. The following matrix elements \( \langle 0 | B_s(p) \rangle \) and \( \langle 0 | J_{\mu}^{V}(p) \rangle \) are defined in terms of the leptonic decay constant and four parameters \( h_{s}^\nu \) as

\[ \langle 0 | B_s(p) \rangle = -i \frac{f_{B_s} m_{D_s}^2}{m_b + m_s}, \]

\[ \langle 0 | J_{\mu}^{V}(p) \rangle = -i \frac{h_{s}^\nu}{2 m_s}, \]

(10)

where correlating the \( h_{s}^\nu \) to \( f_{s} \) and \( f_{q} \), the values \( h_{s}^\nu = -0.053 \text{ GeV}^3 \) and \( h_{q}^\nu = 0.065 \text{ GeV}^3 \) are obtained (for details see [6]). From the Lorentz invariance and parity considerations, the remaining matrix element, i.e., transition matrix element in Eq. (9) is parameterized in terms of form factors in the following way:

\[ \langle P(p') | J_{\mu}^{V}(p) \rangle = \mathcal{P}_\mu f_+(q^2) + q_\mu f_-(q^2), \]

\[ \langle P(p') | J_{\mu}^{T}(p) \rangle = f_T(q^2) \frac{m_B}{m_{B_s} + m_p} \]

(11)

where \( f_+(q^2), f_-(q^2) \) and \( f_T(q^2) \) are the transition form factors, which only depend on the momentum transfer squared \( q^2 \), \( \mathcal{P}_\mu = (p + p')_\mu \) and \( q_\mu = (p - p')_\mu \).

Using Eqs. (10) and (11) in Eq. (9), we obtain

\[ \Pi_{\mu}^{V}(p^2, p'^2, q^2) = \frac{f_{B_s} m_{B_s}^2}{2m_s(m_b + m_s)(m_{B_s}^2 - p^2)} \frac{h_{s}^\nu}{(m_{B_s}^2 - m_p^2)(m_{B_s}^2 - p^2)} \]

\[ \times \left[ f_+(q^2) \mathcal{P}_\mu + f_-(q^2) q_\mu \right]. \]

\[ \Pi_{\mu}^{T}(p^2, p'^2, q^2) = \frac{f_{B_s} m_{B_s}^2}{2m_s(m_b + m_s)(m_{B_s}^2 - p^2)} \frac{h_{s}^\nu}{(m_{B_s}^2 - m_p^2)(m_{B_s}^2 - p^2)} \]

\[ \times \left[ f_T(q^2) \mathcal{P}_\mu - (m_{B_s}^2 - m_p^2) q_\mu \right]. \]

(12)
For extracting the sum rules for form factors $f_+(q^2)$ and $f_-(q^2)$, we choose the coefficients of the structures $P_\mu$ and $q_\mu$ from $\Pi^\per_i(p^2, p'^2, q^2)$, respectively, and the structure $q_\mu$ from $\Pi^\per_i(p^2, p'^2, q^2)$ is considered to calculate the form factor $f_T(q^2)$. Therefore, the correlation functions are written in terms of the selected structures as

$$
\Pi^\per_i(p^2, p'^2, q^2) = \Pi^\per + \Pi^\per q_\mu,
$$

where $i$ stands for $+, −, and T$. The perturbative part is written in terms of the double dispersion integral as

$$
\Pi^\per_i = - \frac{1}{(2\pi)^2} \int ds \int ds' \rho^\per_{i, s, s'}(q^2) + \text{subtraction terms},
$$

where the $\rho^\per_{i, s, s'}(q^2)$ are called spectral densities. To get the spectral densities, we need to evaluate the bare loop diagrams in Fig. 1. Calculating these diagrams via the usual Feynman integrals with the help of the Cutkosky rules, i.e., \( \frac{1}{p^2 - m^2} = -2\pi\delta(p^2 - m^2) \), which implies that all quarks are real, leads to the following spectral densities:

\[
\begin{align*}
\rho^\per_{i, s, s'}(q^2) &= I_0 N_c \{ \Delta + s' - 2m_s^2 + 2m_b m_s + (E_1 + E_2)u \}, \\
\rho^\per_{b, s, s'}(q^2) &= I_0 N_c \{ -\Delta + s' + 2m_s^2 - 2m_b m_s + (E_1 - E_2)u \}, \\
\rho^\per_T(s, s', q^2) &= -I_0 N_c \{ \Delta(m_b - m_s) + s'(m_b - m_b) + 2m_s + 2[m_b(E_1 - E_2) + m_s(E_2 - E_1 - 1)]s' + (E_1 - E_2)(m_b - m_b)u \},
\end{align*}
\]

where

\[
\begin{align*}
I_0(s', s', q^2) &= \frac{1}{4\lambda^{1/2}(s', s', q^2)}, \\
\lambda(s', s', q^2) &= s'^2 + s'^2 + q^4 - 2s'q^2 - 2s'^2 - 2ss', \\
E_1 &= \frac{1}{\lambda(s', s', q^2)} [2s'\Delta - s'u], \\
E_2 &= \frac{1}{\lambda(s', s', q^2)} [2ss' - \Delta u], \\
\Delta &= s + m_b^2 - m_b^2,
\end{align*}
\]

and $N_c = 3$ is the color factor.

Now, we focus our attention to calculating the QCD side of the correlation function. This side is calculated at deep Euclidean space, where $-p^2 \rightarrow \infty$ and $-p'^2 \rightarrow \infty$ via the operator product expansion (OPE). To this aim, we write each $\Pi_i$ function (coefficient of each structure) in terms of the perturbative and nonperturbative parts as

\[
\begin{align*}
\Pi^\per_i(p^2, p'^2, q^2) &= \Pi^\per + \Pi^\per q_\mu, \\
\Pi^\per q_\mu(p^2, p'^2, q^2) &= \Pi^\per q_\mu.
\end{align*}
\]

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\[
\begin{align*}
\Delta &= s + m_b^2 - m_b^2,
\end{align*}
\]

and $N_c = 3$ is the color factor.

For the calculation of the nonperturbative contributions in the QCD side, the condensate terms of OPE are considered. The condensate term of dimension three is related to contribution of quark condensate. Figure 2 shows quark-quark condensate diagrams of dimension three. It should be reminded that the quark condensate are considered only for light quarks, and the heavy quark condensate is suppressed by inverse powers of the heavy quark mass. The contribution of diagram (c) in Fig. 2 is zero since applying the double Borel transformation with respect to both variables $p^2$ and $p'^2$ kills its contribution, because only one variable appears in the denominator in this case. Therefore as dimension three, we consider only diagram (d) in Fig. 2. The dimension four operator in OPE is the gluon-gluon condensate. Our calculations show that in this case, the gluon-gluon condensates contributions are very small in comparison with the quark-quark and quark-gluon condensates contributions, and we can easily ignore their contributions. The next operator is dimension five quark-gluon condensate. The diagrams corresponding to the quark-gluon condensate are presented in Fig. 3. Contributions of the diagrams (e) and (f) vanish for the same reason as diagram (c) in Fig. 2. Therefore, only diagrams (g) and (h) contribute to the nonperturbative part of dimension five. In the QCD sum rule approach, the OPE is truncated at some finite order such that Borel transformations play an important role in this cutting. Mainly, the proper regions of the Borel parameters are adopted by demanding that in the truncated OPE, the condensate term with the highest dimension constitutes a small fraction of the total dispersion integral. In the next section, we will explain how these proper regions are obtained. Hence, we will not consider the condensates with $d \geq 6$ that play a minor role in our calculations.

The explicit expressions of $\Pi^\per_{i, non-per}$, are given in the Appendix.

\[
\begin{align*}
\gamma_\mu(1-\gamma_5) \sigma_\mu(1+\gamma_5), \\
\gamma_\mu(1-\gamma_5) \sigma_\mu(1+\gamma_5)
\end{align*}
\]

\[
\begin{align*}
\gamma_\mu(1-\gamma_5) \sigma_\mu(1+\gamma_5), \\
\gamma_\mu(1-\gamma_5) \sigma_\mu(1+\gamma_5)
\end{align*}
\]
where

\[ M = \text{form factors are derived:} \]

Also the operator \( B \) in Eq. (18) is defined as

\[ B = B_{\rho'}(M_1^2)B_{\rho'}(M_2^2), \]

III. NUMERICAL ANALYSIS

We are now ready to present our numerical analysis of the form factors \( f_+(q^2) \), \( f_-(q^2) \), and \( f_T(q^2) \) and calculate branching fractions and longitudinal lepton polarization asymmetries. In our numerical calculations, we use the following values for input parameters: \( m_s = 0.13 \text{ GeV}, \) \( m_b = 4.8 \text{ GeV}, \) \( m_\eta = (547.51 \pm 0.18) \text{ MeV}, \) \( m_\eta' = (957.78 \pm 0.14) \text{ MeV}, \) \( m_{\eta_c} = (5366.3 \pm 0.6) \text{ MeV} [12], \)

\[ |V_{ub}/V_{cb}| = 0.0385, \quad C_1^{\text{eff}} = -0.313, \quad C_0 = 4.344, \quad C_{10} = -4.669 [13], \quad f_{B_s} = (209 \pm 38) \text{ MeV} [14], \quad m_0^2 = (0.8 \pm 0.2) \text{ GeV}^2, \quad \langle s\bar{s} \rangle = (0.8 \pm 0.2)\langle u\bar{u} \rangle, \quad \text{and} \quad \langle u\bar{u} \rangle = -(0.240 \pm 0.010)^3 \text{ GeV}^3. \]

The sum rules for the form factors contain also four auxiliary parameters, namely, Borel mass squares, \( M_1^2 \) and \( M_2^2 \) and continuum thresholds, \( s_0 \) and \( s'_0 \). These are not physical quantities, so our results should be independent of them. The parameters \( s_0 \) and \( s'_0 \) are not totally arbitrary but they are related to the energy of the first excited states with the same quantum numbers as the interpolating currents. They are determined from the conditions that guarantee the sum rules to have the best stability in the allowed \( M_1^2 \) and \( M_2^2 \) regions. The value of continuum threshold \( s_0 \) calculated from the two-point QCD sum rules are taken to be \( s_0 = (34.2 \pm 2) \text{ GeV}^2 [15]. \) We use also the range, \( (m_\rho + 0.3)^2 \leq s_0 \leq (m_\rho + 0.5)^2 \text{ GeV}^2 \) in the \( P = \eta(\eta') \) channel. The working regions for \( M_1^2 \) and \( M_2^2 \) are determined demanding that not only the contributions of the higher states and continuum are effectively suppressed, but contributions of the higher dimensional operators are also small. Both conditions are satisfied in the regions 12 GeV^2 \leq M_1^2 \leq 22 GeV^2 and 4 GeV^2 \leq M_2^2 \leq 10 GeV^2.

The dependence of the form factors \( f_+, f_- \) and \( f_T \) on \( M_1^2 \) and \( M_2^2 \) for \( B_s \to \eta_s \) transition when \( m_p = m_\eta \) are shown in Fig. 4. Figure 5, also depicts the dependence of the same form factors on Borel mass parameters for \( B_s \to \eta_s \) decay when \( m_p = m_\eta. \) These figures show good stability of the form factors with respect to the Borel mass parameters in the working regions. Using these regions for \( M_1^2 \) and \( M_2^2 \), our numerical analysis shows that the contribution of the nonperturbative part to the QCD side is about 21% of the total, and the main contribution comes from the perturbative part.

Now, we proceed to present the \( q^2 \) dependency of the form factors. Since the form factors \( f_+(q^2) \) and \( f_T(q^2) \) are calculated in the spacelike \( (q^2 < 0) \) region, we should analytically continue them to the timelike \( (q^2 > 0) \) or physical region. Hence, we should change \( q^2 \) to \( -q^2 \). As we previously mentioned, the form factors are truncated at approximately 1 GeV below the perturbative cut. Therefore, to extend our results to the full physical region, we look for parametrization of the form factors in such a way that in the reliable region the results of the parameterization coincide with the sum rules predictions. Our
numerical calculations show that the sufficient parametrization of the form factors with respect to $q^2$ is

$$f_i(q^2) = \frac{f_i(0)}{1 + \alpha \hat{q} + \beta \hat{q}^2},$$

(22)

FIG. 4. The dependence of the form factors on $M_1^2$ and $M_2^2$ for $B_s \rightarrow \eta_s$ decay when $m_P = m_{\eta'}$. The solid, dashed, and dashed-dotted lines correspond to the $f_+, f_-$, and $f_T$, respectively.

FIG. 5. The dependence of the form factors on $M_1^2$ and $M_2^2$ for $B_s \rightarrow \eta_s$ decay when $m_P = m_{\eta'}$. The solid, dashed, and dashed-dotted lines correspond to the $f_+, f_-$, and $f_T$, respectively.

where $\hat{q} = q^2/m_{\eta'}^2$. The values of the parameters $f_i(0)$, $\alpha$, and $\beta$ are given in the Table I taking $M_1^2 = 12$ GeV$^2$ and $M_2^2 = 5$ GeV$^2$. Table I also contains the predictions of the light-front quark model (LFQM). The form factors of $B_s \rightarrow \eta$ and $B_s \rightarrow \eta'$ are obtained using the values in Table I and also Eq. (4).
The values of the form factors at \( q^2 = 0 \) are also compared with the predictions of the other nonperturbative approaches such as, the LFQM and constituent quark model (CQM) in Table II.

Now, we would like to evaluate the branching ratios for the considered decays. Using the parametrization of these transitions in terms of the form factors, we get [17]

\[
\frac{d\Gamma}{dq^2}(B_s \rightarrow P \bar{P}) = \frac{AG_F^2|V_{ts}V_{tb}^*|^2 m_B^3 \alpha^2}{2^8 \pi^5} \frac{|D_\nu(x_r)|^2}{\sin^2 \theta_W} \phi^{3/2}(1, \tilde{r}, \tilde{s})|f_+(q^2)|^2, \\
\frac{d\Gamma}{dq^2}(B_s \rightarrow P l^+ l^-) = \frac{AG_F^2|V_{ts}V_{tb}^*|^2 m_B^3 \alpha^2}{3 \cdot 2^7 \pi^5} \nu \phi^{3/2}(1, \tilde{r}, \tilde{s}) \left[ \left( 1 + \frac{2\tilde{r}}{\tilde{s}} \right) \phi(1, \tilde{r}, \tilde{s}) \alpha_1 + 12\tilde{r} \beta_1 \right],
\]

where \( A = \sin^2 \varphi \) for \( B_s \rightarrow \eta \) and \( A = \cos^2 \varphi \) for \( B_s \rightarrow \eta' \) transitions. The \( \tilde{r}, \tilde{s}, \tilde{l}, x_t \) and \( \tilde{m}_b \) and the functions \( \nu, \phi(1, \tilde{r}, \tilde{s}), D_\nu(x_r) \), \( \alpha_1 \) and \( \beta_1 \) are defined as

\[
\tilde{r} = \frac{m_B^2}{m_{B_s}^2}, \quad \tilde{s} = \frac{q^2}{m_{B_s}^2}, \quad \tilde{l} = \frac{\tilde{m}_b^2}{m_{B_s}^2}, \quad x_t = \frac{m_t^2}{m_{B_s}^2}, \quad \tilde{m}_b = \frac{m_b}{m_{B_s}} \quad \nu = \sqrt{1 - \frac{4\tilde{r}}{\tilde{s}}},
\]

\[
\phi(1, \tilde{r}, \tilde{s}) = 1 + \hat{r}^2 + \tilde{s}^2 - 2\hat{r} \tilde{s} - 2\tilde{r} \hat{r} - 2\tilde{r} \tilde{s}, \quad D_\nu(x_r) = \frac{x_t}{8} \left[ \frac{2 + x_t}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t \right],
\]

\[
\alpha_1 = \left| C_{10}^{\text{eff}} f_+(q^2) + \frac{2\tilde{m}_b C_{10}^{\text{eff}} f_+(q^2)}{1 + \sqrt{\tilde{r}}} \right|^2 + \left| C_{10}^{\text{eff}} f_+(q^2) \right|^2,
\]

\[
\beta_1 = \left| C_{10}^{\text{eff}} \right|^2 \left[ \left( 1 + \tilde{r} \frac{\tilde{s}}{2} \right)|f_+(q^2)|^2 \right]^2 + (1 - \hat{r}) \text{Re}(f_+(q^2)f_+(q^2)) + \frac{1}{2} \tilde{s}|f_-(q^2)|^2 \right] \right].
\]

Integrating Eq. (23) over \( q^2 \) in the whole physical region and using the total mean lifetime \( \tau_{B_s} = (1.466 \pm 0.059) \) ps [12], the branching ratios of the \( B_s \rightarrow (\eta, \eta') l^+ l^- / \bar{r} \bar{v} \) are obtained as presented in Table III. In Table III, we show only the values obtained considering the short-distance (SD) effects contributing to the Wilson coefficient \( C_9^{\text{eff}} \) for the charged lepton case. The effective Wilson coefficient \( C_9^{\text{eff}} \), including both the SD and long distance (LD) effects, is [13]

\[
C_9^{\text{eff}}(s) = C_9 + Y_{SD}(s) + Y_{LD}(s).
\]

<table>
<thead>
<tr>
<th>( B_s \rightarrow \eta_s(P = \eta) )</th>
<th>( B_s \rightarrow \eta_s(P = \eta') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_+(0) )</td>
<td>( -0.2 \pm 0.1 )</td>
</tr>
<tr>
<td>( f_-(0) )</td>
<td>( -0.2 \pm 0.1 )</td>
</tr>
<tr>
<td>( f_T(0) )</td>
<td>( -0.4 \pm 0.1 )</td>
</tr>
</tbody>
</table>

**Table II.** The form factors of the \( B_s \rightarrow \eta_s \) decay for \( M_{T1}^2 = 12 \) GeV\(^2\) and \( M_{T2}^2 = 5 \) GeV\(^2\) at \( q^2 = 0 \) in different approaches: this work (3PSR), the LFQM, and CQM.
TABLE III. The branching ratios in different models corresponding to $\varphi = 41.5^\circ$. The values in parentheses related to $\varphi = 39.3^\circ$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$Br(B_s \to \eta \nu \bar{\nu}) \times 10^6$</td>
<td>$1.4 \pm 0.6$</td>
<td>$1.54$</td>
<td>$2.56(2.34)$</td>
<td>$2.38(2.17)$</td>
<td>$0.95 \pm 0.2$</td>
<td>$2.2 \pm 0.7$</td>
<td>$2.9 \pm 1.5$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta' \nu \bar{\nu}) \times 10^6$</td>
<td>$1.3 \pm 0.6$</td>
<td>$1.47$</td>
<td>$2.36(2.52)$</td>
<td>$2.23(2.38)$</td>
<td>$0.9 \pm 0.2$</td>
<td>$1.9 \pm 0.5$</td>
<td>$2.4 \pm 1.3$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta \mu^+ \mu^-) \times 10^7$</td>
<td>$2.3 \pm 1.0$</td>
<td>$2.09$</td>
<td>$3.75(3.42)$</td>
<td>$3.42(3.12)$</td>
<td>$1.2 \pm 0.3$</td>
<td>$2.6 \pm 0.7$</td>
<td>$3.4 \pm 1.8$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta' \mu^+ \mu^-) \times 10^7$</td>
<td>$2.2 \pm 1.0$</td>
<td>$1.98$</td>
<td>$3.40(3.63)$</td>
<td>$3.19(3.41)$</td>
<td>$1.1 \pm 0.3$</td>
<td>$2.2 \pm 0.6$</td>
<td>$2.8 \pm 1.5$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta \tau^+ \tau^-) \times 10^8$</td>
<td>$3.7 \pm 1.6$</td>
<td>$5.14$</td>
<td>$7.33(6.70)$</td>
<td>$7.33(6.70)$</td>
<td>$3 \pm 0.5$</td>
<td>$8 \pm 1.5$</td>
<td>$10 \pm 5.5$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta' \tau^+ \tau^-) \times 10^8$</td>
<td>$2.8 \pm 1.2$</td>
<td>$2.86$</td>
<td>$4.66(5.00)$</td>
<td>$4.04(4.30)$</td>
<td>$1.55 \pm 0.3$</td>
<td>$3.85 \pm 0.75$</td>
<td>$4.7 \pm 2.5$</td>
</tr>
</tbody>
</table>

TABLE IV. The branching ratios of the semileptonic $B_s \to (\eta, \eta')\mu^+ \mu^-$ decays, including the LD effects.

<table>
<thead>
<tr>
<th>Mode</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Br(B_s \to \eta \mu^+ \mu^-)$</td>
<td>$(1.8 \pm 0.7) \times 10^{-7}$</td>
<td>$(2.2 \pm 0.9) \times 10^{-8}$</td>
<td>$(2.3 \pm 0.9) \times 10^{-8}$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta' \mu^+ \mu^-)$</td>
<td>$(1.8 \pm 0.7) \times 10^{-7}$</td>
<td>$(2.2 \pm 0.9) \times 10^{-8}$</td>
<td>$(1.3 \pm 0.5) \times 10^{-8}$</td>
</tr>
</tbody>
</table>

The LD effect contributions are due to the $J/\psi$ family. The explicit expressions of the $Y_{SD}(s)$ and $Y_{LD}(s)$ can be found in $[13]$ (see also $[18]$). Table III also includes a comparison between our results and the predictions of the other approaches, including the LFQM, CQM, and other methods $[9]$. Note that the results of $[9]$ are not the results directly obtained by analysis of the $B_s \to \eta(\eta')\pi^0$, but they have been found relating the form factors of $B_s \to \eta_s$ to the form factors of $B \to K$ using the quark flavor scheme (see $[9]$). Hence, the comparison of our results with the predictions of $[9]$ is an approximate and for the exact comparison, the form factors should be directly available. In Table III, the set $A$ refers to the values computed using short-distance QCD sum rules, set $B$ shows the results obtained by light-cone QCD sum rules, and set $C$ corresponds to the results calculated via light-cone QCD sum rules within the soft collinear effective theory. From Table III, we see good consistency in the order of magnitude between our results and the predictions of the other nonperturbative approaches. Here, we should also stress that the results obtained for the electron are very close to the results of the muon and for this reason, we only present the branching ratios for muon in our Tables.

In this section, we would like to present the branching ratios, including the LD effects. We introduce some cuts around the resonances of $J/\psi$ and $\psi'$ and study the following three regions for muon:

TABLE V. The branching ratios of the semileptonic $B_s \to (\eta, \eta')\tau^+ \tau^-$ decays, including the LD effects.

<table>
<thead>
<tr>
<th>Mode</th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Br(B_s \to \eta \tau^+ \tau^-)$</td>
<td>$(0.4 \pm 0.2) \times 10^{-9}$</td>
<td>$(3.2 \pm 1.3) \times 10^{-8}$</td>
</tr>
<tr>
<td>$Br(B_s \to \eta' \tau^+ \tau^-)$</td>
<td>$(0.4 \pm 0.2) \times 10^{-9}$</td>
<td>$(2.3 \pm 0.9) \times 10^{-8}$</td>
</tr>
</tbody>
</table>

FIG. 6. The dependence of the differential branching fraction of the $B_s \to \eta \tau^+ \tau^-$ decay with and without the LD effects on $q^2$. The solid and dotted lines show the results without and with the LD effects, respectively.
where $\sqrt{q_{\text{min}}^2} = 2m_\tau$. In Tables IV and V, we present the branching ratios for muon and tau obtained using the regions shown in Eqs. (26) and (27), respectively. The errors presented in Tables III, IV, and V are due to uncertainties in the determination of the auxiliary parameters, errors in input parameters, systematic errors in QCD sum rules as well as the errors associated to the following approximations used in the present work: (a) the form factors are calculated in the low $q^2$ region and extrapolated to high $q^2$ using the fit parametrization in Eq. (22), (b) the hadronic operators in the considered Hamiltonian can receive sizable nonfactorizable corrections, and the
corresponding matrix elements may also be sensitive to the isosinglet content of the \( \eta \) and \( \eta' \) mesons. We show the dependency of the differential branching ratios on \( q^2 \) (with and without LD effects for the charged lepton case) in Figs. 6–11.

Finally, we want to calculate the longitudinal lepton polarization asymmetry for the considered decays. It is given as [17]

\[
P_L = \frac{2\nu}{(1 + \frac{q^2}{m^2})} \phi(1, \hat{r}, \hat{s}) \alpha_1 + 12\beta_1 \\
\times \text{Re} \left[ \phi(1, \hat{r}, \hat{s})(C_7 f T(q^2) - \frac{2C_7 f T(q^2)}{1 + \sqrt{r}})(C_7 f T(q^2))^* \right].
\]

(28)

where \( \nu, \hat{r}, \hat{s}, \phi(1, \hat{r}, \hat{s}), \alpha_1, \) and \( \beta_1 \) were defined before. The dependence of the longitudinal lepton polarization asymmetries for the \( B_s \to (\eta, \eta') l^+ l^- \) decays on the transferred momentum square \( q^2 \) with and without LD effects are plotted in Figs. 12 and 13.

As a result, the order of the obtained values for branching ratios as well as the longitudinal lepton polarization asymmetries show the possibility of studying the considered transitions at LHC. Any experimental measurements on the presented quantities and those comparisons with the obtained results can give valuable information about the nature of the \( \eta \) and \( \eta' \) mesons and strong interactions inside them.

**ACKNOWLEDGMENTS**

The partial support of the Shiraz University Research Council is appreciated.
APPENDIX

In this appendix, the explicit expressions of the $\Pi_{t}^{\text{non-per}}$ are given,

$$\Pi_{t}^{\text{non-per}}(p^2, p^\prime^2, q^2) = \langle \langle s \rangle \rangle \left( \frac{- \frac{m_s}{2rr} + \frac{4m_0^2 m_s - 2m_0^2 m_b + 3m_1^2 - 3m_2^2 m_b}{12rr^2}}{12rr^2} \right. \right.$$

$$+ \frac{m_0^2 m_b q^2 - m_0^2 m_b q^2 + 3m_1^2 m_b - 3m_2^2 m_b + 3m_3^2 m_b^2 - 3m_3^2 q^2 + 2m_0^2 m_s - 4m_0^2 m_b + 3m_2^2 m_b - 3m_2^2 m_b}{12rr^2} \left. \right)$$

$$+ \frac{m_0^2 m_b^2 - 2m_0^2 m_b^2 + 2m_0^2 m_b^2 - m_0^2 m_b^2 m_s + 2m_0^2}{4r^3 r'} \right)$$

$$\Pi_{t}^{\text{non-per}}(p^2, p^\prime^2, q^2) = \langle \langle s \rangle \rangle \left( \frac{2m_0^2 m_b - 9m_1^2 + 3m_1^2 m_b + 3m_1^2 - m_1^2 m_b - 3m_2^2 m_b + 3m_3^2 m_b^2 + 3m_3^2 m_b^2 + 2m_1^2 m_s}{12rr^2} \right.$$

$$+ \frac{3m_2^2 m_b^2 + 3m_1^2 m_b^4 - 3m_2^2 m_b^4 - 3m_1^2 q^2 - 3m_2^2 q^2 + 2m_0^2 m_s - 3m_0^2 m_b + 6m_1^2 m_b - 3m_2^2 m_b}{12rr^2} \left. \right)$$

$$+ \frac{2m_0^2 m_b^2 - m_0^2 m_b^2 + 2m_0^2 m_b^2 - m_0^2 m_b^2 m_s + 2m_0^2}{4r^3 r'} \right)$$

$$\Pi_{t}^{\text{non-per}}(p^2, p^\prime^2, q^2) = \langle \langle s \rangle \rangle \left( \frac{2m_1^2 + 2m_1 m_b + m_0^2}{4r^2 r'} + \frac{m_1^2 - m_0^2 m_b + 3m_0^2 q^2 - m_0^2 q^2 + 3m_1^2 - 6m_1^2 m_b - m_0^2 m_b}{12rr^2} \right.$$

$$+ \frac{m_0^2 m_b^2 - m_0^2 m_b^2 + 3m_0^2 - 3m_0^2 m_b^2 - 3m_0^2 m_b^2 + m_0^2 m_b^2 m_s}{6r^2 r'} + \frac{m_0^2 m_b^2 q^2 - m_0^2 m_b^2 q^2 + 3m_0^2 m_b^2 q^2 - 3m_0^2 q^2}{6r^2 r'}$$

$$+ \frac{m_0^2 m_s - m_0^2 m_b + 3m_0^2 q^2 - 3m_0^2 m_b^2 + 3m_0^2 q^2 + m_0^2 m_b m_s - 3m_1^2 m_b}{6r^2 r'}$$

$$+ \frac{m_0^2 m_b^2 - 2m_0^2 m_b^2 + 2m_0^2 m_b^2 - m_0^2 m_b^2 m_s + 2m_0^2}{2r^3 r'} \right)$$

where $r = p^2 - m_s^2$ and $r' = p^\prime^2 - m_b^2$. 

FIG. 13. The same as Fig. 12 but for the $B_s \rightarrow \eta'$ transition.
RARE SEMILEPTONIC $B_s$ DECAYS TO $\eta$ AND …